SUPERMODULARITY
AND COMPLEMENTARITY

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Supermodularity and Complementarity

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dedicated
with love
to my children

Adam and Naomi

Contents

Preface

This monograph has had a remarkably long gestation period. My involvement in this area spans three decades, commencing with my doctoral dissertation (Topkis [1968]), which studied the problem of maximizing a supermodular function on a lattice with its implications for monotone comparative statics and with applications to various decision problems. The study of complementarity, of course, goes back much further. Yet a research monograph on the present subject—covering supermodular functions on a lattice, their connection with complementarity, their role for monotone comparative statics, and their use in diverse economic models—is certainly timely. Supermodularity has received considerable attention in the economics literature in recent years, with important extensions of the theory and significant new applications. However, much of the previously published theory and its applications were developed primarily in the applied mathematics and operations research literature, and the basic concepts and results are relatively new to the economics lexicon. This monograph is intended to present a self-contained and up-to-date view of this field, including many new results, to scholars interested in economic theory and its applications as well as to those in related disciplines. The objectives are to provide an integrated and accessible view of the relevant theory pertaining to supermodularity, complementarity, and monotone comparative statics, and to demonstrate the utility of the methodology through diverse applications of that theory to economic problems. I am grateful to Kyle Bagwell, Hervé Moulin, Tom Sargent, and Bill Sharkey for helpful comments on earlier versions of this monograph.

SUPERMODULARITY AND COMPLEMENTARITY

1

Introduction

One may sometimes conclude too readily that an old, familiar, and simple concept, exemplified in myriad common situations, has little new to offer. In particular, how much novelty could be expected from the descriptive notion of complementarity, whereby two products are considered complements if having more of one product increases the marginal value derived from having more of the other product? Indeed, a half century ago, Samuelson [1947] declared:

> In my opinion the problem of complementarity has received more attention than is merited by its intrinsic importance.

Yet, a quarter century later, Samuelson [1974] came to assert:

> The time is ripe for a fresh, modern look at the concept of complementarity. . . . [T]he last word has not yet been said on this ancient preoccupation of literary and mathematical economists. The simplest things are often the most complicated to understand fully.

The latter quotation also expresses the motivation for this monograph, which links complementarity to powerful tools involving supermodular functions on lattices and focuses on analyses and issues related to monotone comparative statics.

Comparative statics examines how optimal decisions or equilibria in a parameterized collection of problems vary with respect to the parameter. In a decision problem, the parameter may affect the objective function and the feasible region. In a noncooperative game, the parameter may affect the payoff functions of the various players, the collection of feasible strategies for the players, and the set of players participating in the game. In a cooperative game, the parameter may affect the characteristic function and the set of players in the game. Monotone comparative statics is particularly concerned with scenarios where optimal decisions or equilibria vary monotonically with the parameter. Several alternate notions of complementarity may be stated in terms of monotone comparative statics. One notion would consider a system of products to be complements if, treating the levels of any subset of the products as a parameter, optimal levels for the other products increase with the parametric levels of the former subset of products. Another notion would consider a system of products to be complements if the acquisition prices for the products are treated as a parameter and the optimal levels of all products increase as the prices of the products decrease. These two alternate notions of complementarity turn out to be very closely related to that notion given in the second sentence of the preceding paragraph. The key to these relationships is found in properties of supermodular functions, which are intimately related to complementarity.

The economics literature is replete with examples of monotone comparative statics. Most of these examples are manifestations of complementarity, with a common explicit or implicit theoretical basis in properties of a supermodular function on a lattice. Supermodular functions on a lattice yield a characterization for complementarity and extend the notion of complementarity to a general and abstract setting, which is a natural mathematical context for studying complementarity and monotone comparative statics. Within this context, handy analytical tools are available and may be further developed. Supermodularity is a basis of a unifying theory for analyzing structural properties of a collection of parameterized optimization problems. Fundamental results give sufficient and necessary conditions under which optimal solutions for a collection of parameterized optimization problems vary monotonically with the parameter. Concepts and results related to supermodular functions on a lattice constitute a formal step in a long line of economics literature on complementarity. While informal statements of properties related to complementarity are intuitively appealing, this monograph cites several anomalies where examples with some flavor of complementarity fail to possess properties that intuition might expect.

This monograph develops a comprehensive theory relating to supermodular functions on a lattice, with a focus on the connection between supermodular functions and complementarity as well as on the role of supermodularity and complementarity for monotone comparative statics. Furthermore, this monograph exhibits the use of that theory in the analysis of many diverse models. The emphasis is on methodology. The theoretical material is a systematic and integrated view that summarizes and refines previously published research, fills in gaps, makes new connections, and introduces new results. The present framework makes available tools with which one can construct concise proofs having to do with supermodularity, complementarity, and monotone comparative statics, rather than deriving ad hoc proofs for different models. It facilitates identifying the most general scenarios and assumptions for models to exhibit monotone comparative statics and other properties related to complementarity, as well as delimiting classes of models and structural hypotheses for which such properties might hold. And it serves to elucidate a common theoretical basis whereby apparently disparate models share essential features and to enable the recognition of essential distinctions between models. The choice of theoretical material for inclusion here is based on potential relevance and utility and on providing a relatively complete perspective of salient issues. The inclusion of examples and applications, some previously published and some new, is far more selective than the theoretical material. Besides their intrinsic importance, the particular selection of examples and applications is based on how well they illustrate uses of the present methodology, where this methodology constitutes the primary technical issue and is not clouded by substantial extraneous technical issues. The exposition of the theory and the applications is, for the most part, self-contained.

The heart of this monograph is Chapter 2, which presents concepts and theory relating to lattices, supermodularity, complementarity, and monotone comparative statics. It develops tools to facilitate the analysis of supermodularity, complementarity, and monotone comparative statics, and it clarifies the role of lattices, supermodularity, and complementarity for monotone comparative statics. Topics in Chapter 2 include partially ordered sets and lattices; completeness and related topological properties; an ordering relation used to compare sets of feasible solutions or sets of optimal solutions; fixed points; supermodular functions and increasing differences; maximization of supermodular functions; sufficient and necessary conditions for monotone comparative statics; and equivalences and implications among various notions related to complementarity. The theory does without traditional assumptions of differentiability, concavity, and divisibility that are required by classical methods for comparative statics. Some brief illustrative examples are included.

Chapter 3, Chapter 4, and Chapter 5 apply theory from Chapter 2 to decision problems, noncooperative games, and cooperative games, respectively. These three applications chapters are largely independent of one another. Chapter 3 studies matching workers to firms; a detailed model of a firm engaged in manufacturing and marketing operations; production planning; transportation and transshipment problems; other network models (the shortest path problem in an acyclic network, the minimum cut problem, the maximum closure problem) with applications to dynamic economic lot size production models, the selection of activities, and equivalences among certain combinatorial structures; the optimality of myopic decisions; Markov decision processes; stochastic inventory problems; and stochastic transformations related to the latter two classes of decision problems. Chapter 4 examines the existence of an equilibrium point in a noncooperative game; parametric properties of equilibrium points; algorithms for approximating an equilibrium point; and examples. Chapter 5 looks at the core of a cooperative game with side payments; the greedy algorithm; cooperative games depending on a parameter; a class of activity optimization games; and examples.

2

Lattices, Supermodular Functions, and Related Topics

2.1 Introduction

This chapter includes general concepts and results relevant for the present perspective on supermodularity and complementarity. The theory in this chapter is used in the applications that follow in the three subsequent chapters. For a collection of optimization problems where the objective function and the constraint set depend on a parameter, **comparative statics** is concerned with the dependence of optimal solutions on the parameter and **monotone comparative statics** is concerned with optimal solutions varying monotonically with the parameter. (These definitions extend from optimization problems to game problems, as in Chapter 4 and Chapter 5.) Monotone comparative statics is the primary issue considered herein with respect to supermodularity and complementarity.

For the problem of maximizing (or minimizing) a real-valued $f(x)$ over x in a set X, the function $f(x)$ is the **objective function**, x is the **decision variable**, the set X of feasible values for the decision variable x is the **constraint set**, and $\mathbf{argmax}_{x \in X} f(x)$ and $\mathbf{argmax}\{f(x) : x \in X\}$ both denote the set of all optimal solutions (that is, $\{x' : x' \in X, f(x) \leq f(x') \text{ for each } x \in X\}$). Likewise, define $\mathbf{argmin}_{x \in X} f(x)$ and $\mathbf{argmin}\{f(x) : x \in X\}$ as $\text{argmax}_{x \in X}(-f(x))$. The focus in monotone comparative statics involves the collection of parameterized optimization problems

$$\text{maximize } f(x, t) \text{ subject to } x \in S_t \tag{2.1.1}$$

for each **parameter** t contained in a parameter set T, where x is the **decision variable** contained in a set X, the **constraint set** S_t is a subset of X and depends on t, and the **objective function** $f(x, t)$ depends upon t. The parameter t from T is then reflected in the notation for the set of optimal solutions as $\text{argmax}_{x \in S_t} f(x, t)$ and $\text{argmax}\{f(x, t) : x \in S_t\}$. Monotone comparative statics considers conditions under which the set of optimal solutions $\text{argmax}_{x \in S_t} f(x, t)$ is, in a sense, increasing in the parameter t and one can select an optimal solution x_t in $\text{argmax}_{x \in S_t} f(x, t)$ for each t in T such that x_t is an increasing function of t (that is, $t' \leq t''$ in T implies $x_{t'} \leq x_{t''}$).

Example 2.1.1, Example 2.1.2, and Example 2.1.3 below present illustrative applications developed further in subsequent sections of this chapter. Each example involves optimally determining the consumption levels of a set of products. In Example 2.1.1, the consumption levels of a subset of the products are the decision variables with the consumption levels of the other products being the parameter. In Example 2.1.2, consumption levels of all products are the decision variables and their prices are the parameter. In Example 2.1.3, consumption levels of all available products are the decision variables with the parameter being the set of products available in the market.

For an n-vector $x = (x_1, \ldots, x_n)$ and a subset I of $\{1, \ldots, n\}$, let $\boldsymbol{x_I} = \{x_i : i \in I\}$. For sets X' and X'', $\boldsymbol{X' \setminus X''}$ denotes the set of elements in X' that are not in X''. For a finite set X, $\boldsymbol{|X|}$ denotes the number of elements in X. The **inner product** of two n-vectors of real numbers, $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_n)$, is $\boldsymbol{t \cdot x} = \sum_{i=1}^{n} t_i x_i$. (The definition here ignores the usual convention that the first vector be a row vector and the second vector be a column vector. Throughout, usual conventions that distinguish between a row vector and a column vector are ignored.) Besides the real number 0, let the symbol **0** also denote a vector with each component being 0 where the dimension of this 0 vector is apparent from context.

Example 2.1.1. A system has n products, $i = 1, \ldots, n$. The consumption level of product i is x_i, and $x = (x_1, \ldots, x_n)$. The vector x must be contained in a subset X of R^n. For x in X, the consumer receives a real-valued utility $f(x)$. Let $N = \{1, \ldots, n\}$. Suppose that the set of products N is partitioned into two sets I and $N \setminus I$, where I is a subset of N and $1 \leq |I| \leq n - 1$. The consumer takes the values $x_{N \setminus I} = \{x_i : i \in N \setminus I\}$ for the consumption levels of the products $N \setminus I$ as given and wants to maximize utility over the consumption levels $x_I = \{x_i : i \in I\}$ of the products I. The decision problem, given $x_{N \setminus I}$, is to maximize the utility $f(x_I, x_{N \setminus I})$ over x_I such that $x = (x_I, x_{N \setminus I})$ is in X. An issue for monotone comparative statics is to provide conditions implying that, for each subset I of N with $1 \leq |I| \leq n - 1$, optimal consumption levels for products I increase when the consumption levels of the other products $N \setminus I$ increase; that is, optimal x_I increases with $x_{N \setminus I}$ for any nonempty strict subset I of the products.

Example 2.1.2. Consider a system with n products, denoted $i = 1, \ldots, n$, where a consumer chooses the consumption level x_i of each product i. Let $x = (x_1, \ldots, x_n)$ be the n-vector of consumption levels for all n products.

The vector x must be contained in a subset X of R^n. The utility of a consumption vector x is given by a real-valued utility function $f(x)$. To acquire (or produce) each product i, there is a unit price p_i. The price vector is $p = (p_1, \ldots, p_n)$. The decision problem is to maximize the net value $f(x) - p \cdot x$, the utility minus the acquisition cost, over x in X. An issue for monotone comparative statics is to provide conditions implying that optimal consumption levels of all products increase when the prices of any subset of products decrease; that is, optimal x increases with $-p$.

Example 2.1.3. Some subset of a collection of n products, denoted $i = 1, \ldots, n$, is available to a consumer in the market. The level of product i consumed is x_i, and $x = (x_1, \ldots, x_n)$. The consumption vector x must be contained in a subset X of R^n. It is feasible to consume nothing, so the 0 vector is in X. For x in X, the consumer receives a real-valued utility $f(x)$. The collection of all n products that may potentially be available in the market is $N = \{1, \ldots, n\}$. Only a subset I of the set of all products N is actually available in the market. The decision problem, given I, is to maximize the utility $f(x_I, x_{N \setminus I})$ over x_I such that $x = (x_I, x_{N \setminus I})$ is in X and $x_{N \setminus I} = 0$. An issue for monotone comparative statics is to provide conditions implying that optimal consumption levels x of all n products increase when the subset I of available products is greater; that is, as additional products become available in the market, it is optimal to consume more of all products.

The sections of this chapter are organized as follows. These sections are logically ordered with some but not all material from each section being used in subsequent sections (except for Section 2.5, which is subsequently used only in Chapter 4).

Section 2.2 introduces some standard concepts involving order and lattices, as well as basic useful properties of lattices, characterizations of sublattices of R^n (and generalizations thereof), and characterizations and examples of those systems of inequality constraints that determine sublattices of R^n and of more general lattices. Ordering is fundamental throughout this monograph. Ordering is used for comparing elements of a parameter set, for comparing selections of feasible or optimal solutions, and for certain properties of the objective function $f(x, t)$ in (2.1.1). Lattices and sublattices also play a fundamental role. The constraint sets of optimal decision problems are subsequently assumed to be sublattices, and the sets of optimal solutions $\operatorname{argmax}_{x \in S_t} f(x, t)$ in (2.1.1) turn out to be sublattices.

Section 2.3 considers complete lattices and subcomplete sublattices, for which each nonempty subset has a least upper bound and a greatest lower bound. These lattices and sublattices are characterized by compactness conditions in R^n and in other topological spaces. The characterizations are useful in establishing conditions for sets of feasible solutions to optimal decision problems and for sets of optimal solutions $\text{argmax}_{x \in S_t} f(x, t)$ in (2.1.1) to have a greatest element and a least element. The existence of these greatest (least) elements is useful in selecting particular elements from those sets.

Section 2.4 provides properties and characterizations of a natural ordering relation, the induced set ordering, on the collection of nonempty sublattices of a given lattice. This ordering relation and its properties are used in the monotone comparative statics conditions of Section 2.8 for ordering the sets of feasible solutions in parameterized decision problems and the sets of optimal solutions $\text{argmax}_{x \in S_t} f(x, t)$ in (2.1.1), as well as for selecting specific ordered elements from these sets.

Section 2.5 gives conditions for a function or a correspondence on a lattice to have a fixed point, and for a parameterized collection of functions or correspondences on a lattice to have a fixed point for each parameter value such that the fixed point is an increasing function of the parameter. These results are applied in Chapter 4 to establish the existence and monotonicity of equilibrium points in certain noncooperative games.

Section 2.6 introduces supermodular functions on a lattice and explores some of their basic properties. There is an equivalence between supermodularity and a standard notion of complementarity, increasing differences. Supermodularity and increasing differences are the essential properties for the objective function in (2.1.1) that are part of the conditions subsequently developed for monotone comparative statics. Various common functional transformations maintain or generate supermodularity. Ordinal generalizations extend the cardinal properties of supermodularity and increasing differences. The specialized notion of log-supermodularity, related to the monotonicity of price elasticity, is also considered.

Section 2.7 examines the problem of maximizing a supermodular function on a lattice. The set of maximizers of a supermodular function is a sublattice, and a consequence is that a greatest maximizer and a least maximizer exist under standard regularity conditions. Conversely, the set of maximizers of a function being a sublattice yields, in a sense, a characterization of supermodular functions. Another key property, applied repeatedly in subsequent chapters, is that supermodularity is preserved under the maximization operation. This property gives conditions for the optimal value of the objective function in (2.1.1) to be a supermodular function of the parameter t.

Section 2.8 develops fundamental conditions for monotone comparative statics; that is, for the optimal solution sets $\text{argmax}_{x \in S_t} f(x, t)$ in (2.1.1) to be increasing in the parameter t (with respect to the induced set ordering relation from Section 2.4) and for the existence of optimal solutions x_t (in particular, the greatest element or the least element) from $\text{argmax}_{x \in S_t} f(x, t)$ for each t such that x_t is increasing in the parameter t. Sufficient conditions are based on supermodularity and increasing differences and related properties. Conversely, statements of necessity are likewise based on these and closely related properties.

Section 2.9 informally summarizes implications and equivalences between various notions of complementarity encountered in previous sections of this chapter.

While convex sets and concave functions are formally quite different from lattices and supermodular functions, respectively, there nevertheless exist a number of properties holding with similar statements for both the former and the latter. This chapter notes many instances where lattices and supermodular functions have properties analogous to properties of convex sets and concave functions.

2.2 Partially Ordered Sets and Lattices

This section introduces and develops concepts and properties involving order and lattices. Subsection 2.2.1 provides definitions, notation, and basic properties. Subsection 2.2.2 gives characterizations of sublattice structure.

2.2.1 Definitions, Notation, and Some Basic Properties

A **binary relation** $\preceq$ on a set X specifies for all x' and x'' in X either that $x' \preceq x''$ is true or that $x' \preceq x''$ is false. If $x' \preceq x''$ and $x' \neq x''$, then $x' \prec x''$ is written. A binary relation $\preceq$ on a set X is **reflexive** if $x \preceq x$ for each x in X, **antisymmetric** if $x' \preceq x''$ and $x'' \preceq x'$ imply $x' = x''$ for all x' and x'' in X, and **transitive** if $x' \preceq x''$ and $x'' \preceq x'''$ imply $x' \preceq x'''$ for all x', x'', and x''' in X. A **partially ordered set** is a set X on which there is a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. If X is a partially ordered set with binary relation $\preceq$, then the **dual** is the partially ordered set consisting of the same set X with the binary relation $\preceq'$ where $x' \preceq' x''$ for x' and x'' in X if and only if $x'' \preceq x'$. If x' and x'' are elements of a partially ordered set X, $x' \prec x''$, and there does not exist x''' in X with $x' \prec x''' \prec x''$, then x'' **covers** x' in X. Two elements x' and x'' of a partially ordered set are **ordered** if either $x' \preceq x''$ or $x'' \preceq x'$; otherwise, x' and x'' are **unordered**. A partially ordered set is a **chain** if it does not contain an unordered pair of elements.

Example 2.2.1 gives some common examples of partially ordered sets.

Example 2.2.1. The following are partially ordered sets.
(a) The real line $\boldsymbol{R^1}$ with the usual ordering relation $\boldsymbol{\leq}$ on the real numbers is a partially ordered set.
(b) The set $\boldsymbol{R^n} = \{x = (x_1, \ldots, x_n) : x_i \in R^1 \text{ for } i = 1, \ldots, n\}$ with the ordering relation $\boldsymbol{\leq}$ where $x' \leq x''$ in R^n if $x'_i \leq x''_i$ in R^1 for $i = 1, \ldots, n$ is a partially ordered set.
(c) If X_α is a set for each α in a set A, then the **direct product** of these sets X_α is the **product set** $\boldsymbol{\times_{\alpha \in A} X_\alpha} = \{x = (x_\alpha : \alpha \in A) : x_\alpha \in X_\alpha$ for each $\alpha \in A\}$. (The vector $(x_\alpha : \alpha \in A)$ consists of a component x_α for each α in A.) If X_α is a partially ordered set with binary relation $\preceq_\alpha$ for each α in A, then the **direct product** of these partially ordered sets is the partially ordered set consisting of the set $\times_{\alpha \in A} X_\alpha$ with the **product relation** $\preceq$ where $x' \preceq x''$ in $\times_{\alpha \in A} X_\alpha$ if $x'_\alpha \preceq_\alpha x''_\alpha$ for each α in A. Part (b) above is a special case of this direct product example, where $A = \{1, \ldots, n\}$, $X_\alpha = R^1$ with R^1 having the usual ordering relation $\leq$ for each α in A, and $R^n = \times_{\alpha \in A} X_\alpha$.
(d) The **power set**, $\boldsymbol{\mathcal{P}(X)}$, of a set X is the set of all subsets of X. The power set $\mathcal{P}(X)$ with the **set inclusion** ordering relation $\boldsymbol{\subseteq}$ is a partially ordered set. If X' and X'' are distinct subsets of X with $X' \subseteq X''$, then $\boldsymbol{X' \subset X''}$. This example is equivalent to a case of the direct product in part (c) above, in the following sense. Associate each X' in $\mathcal{P}(X)$ with an **indicator vector** $\boldsymbol{1(X')}$ $= (1(X')_x : x \in X)$, where $1(X')_x = 1$ if x is in X' and $1(X')_x = 0$ if x is not in X'. Then $X' \subseteq X''$ in $\mathcal{P}(X)$ if and only if $1(X') \preceq 1(X'')$ in $\times_{x \in X}\{0, 1\}$, where $\{0, 1\}$ has the usual ordering relation $\leq$.
(e) The **lexicographic ordering** relation $\boldsymbol{\leq_{lex}}$ on R^n is such that $x' \leq_{lex} x''$ in R^n if either $x' = x''$ or there is some i' with $1 \leq i' \leq n$, $x'_i = x''_i$ for each i with $1 \leq i < i'$, and $x'_{i'} < x''_{i'}$. The set R^n with the lexicographic ordering relation $\leq_{lex}$ is a partially ordered set; indeed, it is a chain.

Unless explicitly stated otherwise in a particular context, the following conventions are used throughout this monograph. Whenever discussing a general partially ordered set, the associated ordering relation is denoted $\preceq$. Sometimes, as in the definition of "increasing" below, the same symbol $\preceq$ may be used to denote different ordering relations on different partially ordered sets, where the particular context precludes any ambiguities. Any subset of R^n is taken to have $\leq$ as the associated ordering relation. A subset of the direct product of partially ordered sets is taken to have the product relation as the associated ordering relation.

Suppose that X is a partially ordered set and X' is a subset of X. If x' is in X and $x \preceq x'$ ($x' \preceq x$) for each x in X', then x' is an **upper (lower) bound** for X'. If x' in X' is an upper (lower) bound for X', then x' is the **greatest (least)** element of X'. If x' is in X' and there does not exist any x'' in X' with $x' \prec x''$ ($x'' \prec x'$), then x' is a **maximal** (**minimal**) element of X'. A greatest (least) element is a maximal (minimal) element. A partially ordered set can have at most one greatest (least) element, but it may have any number of maximal (minimal) elements. Distinct maximal (minimal) elements are unordered. If the set of upper (lower) bounds of X' has a least (greatest) element, then this **least upper bound** (**greatest lower bound**) of X' is the **supremum** (**infimum**) of X' and is denoted $\mathbf{sup}_X(X')$ ($\mathbf{inf}_X(X')$) if the set X is not clear from context or $\mathbf{sup}(X')$ ($\mathbf{inf}(X')$) if the set X is clear from context. One must be clear about the underlying set X in expressing $\sup(X')$ ($\inf(X')$), as the following example shows.

Example 2.2.2. Suppose that $X = R^1$, $Y = [0, 1) \cup \{2\}$, and $X' = [0, 1)$. Then $\sup_X(X') = 1 \neq 2 = \sup_Y(X')$.

If two elements, x' and x'', of a partially ordered set X have a least upper bound (greatest lower bound) in X, it is their **join** (**meet**) and is denoted $\boldsymbol{x' \vee x''}$ ($\boldsymbol{x' \wedge x''}$). A partially ordered set that contains the join and the meet of each pair of its elements is a **lattice**. Example 2.2.3 gives some basic, useful examples of lattices.

Example 2.2.3. The following are lattices.
(a) The real line R^1 is a lattice with $x' \vee x'' = \max\{x', x''\}$ and $x' \wedge x'' = \min\{x', x''\}$ for x' and x'' in R^1.
(b) Any chain is a lattice.
(c) For any positive integer n, R^n is a lattice with $x' \vee x'' = (x'_1 \vee x''_1, \ldots, x'_n \vee x''_n)$ and $x' \wedge x'' = (x'_1 \wedge x''_1, \ldots, x'_n \wedge x''_n)$ for x' and x'' in R^n.
(d) The direct product of lattices is a lattice. If X_α is a lattice for each α in A, $x' = (x'_\alpha : \alpha \in A)$ and $x'' = (x''_\alpha : \alpha \in A)$ are in $\times_{\alpha \in A} X_\alpha$, and $x'_\alpha \vee x''_\alpha$ and $x'_\alpha \wedge x''_\alpha$ are the join and meet of x'_α and x''_α in X_α for each α in A, then $x' \vee x'' = (x'_\alpha \vee x''_\alpha : \alpha \in A)$ and $x' \wedge x'' = (x'_\alpha \wedge x''_\alpha : \alpha \in A)$ in $\times_{\alpha \in A} X_\alpha$.
(e) For any set X, the power set $\mathcal{P}(X)$ with the set inclusion ordering relation $\subseteq$ is a lattice with $X' \vee X'' = X' \cup X''$ and $X' \wedge X'' = X' \cap X''$ for X' and X'' in $\mathcal{P}(X)$. That is, the join of two subsets of X is their union and the meet of two subsets of X is their intersection.

If X' is a subset of a lattice X and X' contains the join and meet (with respect to X) of each pair of elements of X', then X' is a **sublattice** of X. For a lattice X, let $\mathcal{L}(X)$ denote the set of all nonempty sublattices of X. If X' is a sublattice of a lattice X, then X' is itself a lattice and in X' the join and meet of any two elements are the same as the join and meet of those same two elements in X. If X is a lattice, X' is a sublattice of X, and X'' is a sublattice of X', then X'' is a sublattice of X. If X and X' are lattices with the same ordering relation and X' is a subset of X, X' is not necessarily a sublattice of X as the following example shows.

Example 2.2.4. Let $X = R^2$ and $X' = \{(0,0),(2,1),(1,2),(3,3)\}$. Then X' is a lattice, but X' is not a sublattice of X. In X, $(2,1) \vee (1,2) = \sup_X\{(2,1),(1,2)\} = (2,2)$ (which is not in X') and $(2,1) \wedge (1,2) = \inf_X\{(2,1),(1,2)\} = (1,1)$ (which is not in X'). But in X', $(2,1) \vee (1,2) = \sup_{X'}\{(2,1),(1,2)\} = (3,3)$ and $(2,1) \wedge (1,2) = \inf_{X'}\{(2,1),(1,2)\} = (0,0)$.

Example 2.2.5 presents some common sublattices.

Example 2.2.5. The following are sublattices.
(a) If $X = R^1$, then any subset of X is a sublattice of X.
(b) If X is a chain, then any subset of X is a sublattice of X.
(c) If a subset X of R^n has the property that $x' = (x'_1, \dots, x'_n)$ in X and $x'' = (x''_1, \dots, x''_n)$ in X imply that $(\max\{x'_1, x''_1\}, \dots, \max\{x'_n, x''_n\})$ and $(\min\{x'_1, x''_1\}, \dots, \min\{x'_n, x''_n\})$ are in X, then X is a sublattice of R^n.
(d) If X_α is a lattice and X'_α is a sublattice of X_α for each α in a set A, then $\times_{\alpha \in A} X'_\alpha$ is a sublattice of $\times_{\alpha \in A} X_\alpha$.
(e) If X is a lattice, then $\boldsymbol{[x', \infty)} = \{x : x \in X, x' \preceq x\}$ is a sublattice of X for each x' in X, $\boldsymbol{(-\infty, x']} = \{x : x \in X, x \preceq x'\}$ is a sublattice of X for each x' in X, and $\boldsymbol{[x', x'']} = \{x : x \in X, x' \preceq x, x \preceq x''\}$ is a sublattice of X for all x' and x'' in X. These are the **closed intervals** in X.

Since every subset of R^1 is a sublattice of R^1 as in part (a) of Example 2.2.5, sublattices of R^n are not generally endowed with strong topological properties as is the case for convex sets (Rockafellar [1970]). However, the closure of a sublattice of R^n is also a sublattice, and, likewise, the closure of a convex subset of R^n is also convex (Rockafellar [1970]). Other topological properties are discussed in Section 2.3.

Example 2.2.6 shows that those vectors satisfying a budget constraint do not form a sublattice of R^n for $n \geq 2$, but a simple transformation of the variables yields a sublattice. (See Lemma 2.2.7.) Let $\boldsymbol{u^i}$ denote the $\boldsymbol{i^{th}}$ **unit vector** in R^n; that is, for $1 \leq i \leq n$, $u^i = (u^i_1, \dots, u^i_n)$ where $u^i_i = 1$ and $u^i_{i'} = 0$ for each $i' \neq i$ with $1 \leq i' \leq n$.

Example 2.2.6. Suppose that $B > 0$ and n is an integer with $n \geq 2$, and let $X = \{x : x \in R^n, 0 \leq x, \sum_{i=1}^n x_i \leq B\}$. Pick any distinct integers i' and i'' between 1 and n. Then $Bu^{i'}$ and $Bu^{i''}$ are in X, but $Bu^{i'} \vee Bu^{i''} = Bu^{i'} + Bu^{i''}$ is not in X and so X is not a sublattice of R^n. Now consider the transformation $z_{i'} = \sum_{i=1}^{i'} x_i$ for $i' = 1, \dots, n$, and let $Z = \{z : z \in R^n, 0 \leq z_1 \leq \cdots \leq$

$z_n \le B\}$. There is a one-to-one relationship between elements of X and Z, and the set Z is a sublattice of R^n. (See part (b) of Example 2.2.7.)

A function $f(x)$ from a partially ordered set X to a partially ordered set Y is **increasing** (**decreasing**) if $x' \preceq x''$ in X implies $f(x') \preceq f(x'')$ $(f(x'') \preceq f(x'))$ in Y. A function is **monotone** if it is either increasing or decreasing. A function $f(x)$ from a partially ordered set X to a partially ordered set Y is **strictly increasing** (**strictly decreasing**) if $x' \prec x''$ in X implies $f(x') \prec f(x'')$ $(f(x'') \prec f(x'))$ in Y. It is common in the lattice theory literature (Birkhoff [1967]; Topkis [1978]), to use the terms **isotone** and **antitone** rather than "increasing" and "decreasing", respectively, but the latter are used herein in order to maintain a more familiar terminology. The term "increasing" is sometimes used to connote what is here defined as "strictly increasing", but here constant functions fit the definition of "increasing". The term "nondecreasing" is not used instead of "increasing" and "isotone" because it connotes too weak a meaning (that is, "does not increase") when the range of the function is not a chain and would seem to include inappropriately, for example, the function $f(x)$ from $\{0, 1\}$ to R^2 where $f(0) = (1, 2)$ and $f(1) = (2, 1)$.

If $f(x)$ is a function from a set X into a partially ordered set Y, then the **level sets** of $f(x)$ on X are the sets $\{x : x \in X, y \preceq f(x)\}$ for y in Y. A function $f(x)$ from a set X into a partially ordered set Y is a **generalized indicator function** for a subset X' of X if

$$f(x) = \begin{cases} y'' & \text{for } x \in X' \\ y' & \text{for } x \in X \text{ and } x \notin X' \end{cases}$$

where $y' \prec y''$ in Y; that is, if the only level sets of $f(x)$ on X are X and X' and perhaps the empty set. An **indicator function** is a generalized indicator function with $Y = R^1$, $y' = 0$, and $y'' = 1$. An indicator function is equivalent to an indicator vector, as defined in part (d) of Example 2.2.1.

If X is a partially ordered set, X' is a subset of X, and $X \cap [x, \infty)$ is a subset of X' for each x in X', then X' is an **increasing set**. Equivalently, a subset X' of a partially ordered set X is an increasing set if the indicator function of $X' \cap [x, \infty)$ is an increasing function on X for each x in X'. Increasing sets are useful in characterizing properties of parameterized collections of distribution functions, as discussed in Subsection 3.9.1.

Under certain regularity conditions, a nonempty lattice has a greatest element and a least element. This property is useful in providing a way to systematically select an element from a lattice or from each of a collection of lattices. (See Theorem 2.4.3, Corollary 2.7.1, and part (a) of Theorem 2.8.3.) In the context of the present monograph, the set of solutions for an optimization problem is commonly a (sub)lattice (Theorem 2.7.1), and then one may systematically select the greatest (or least) optimal solution. This facilitates comparing particular optimal solutions in stating results about monotone comparative statics. Lemma 2.2.1 shows that a nonempty finite lattice has a greatest element and a least element. An alternate proof of this result comes from combining the properties that any nonempty finite partially ordered set has a maximal element and a minimal element and that a maximal (minimal) element of a lattice is the greatest (least) element. Furthermore, a nonempty compact sublattice of R^n has a greatest element and a least element (Corollary 2.3.2). An alternate proof of that conclusion comes from combining the properties that any nonempty compact subset of R^n has a maximal element and a minimal element (which can be constructed by maximizing or minimizing any strictly increasing continuous function over the nonempty compact set) and that a maximal (minimal) element of a lattice is the greatest (least) element. However, a nonempty sublattice of R^n does not always have a greatest (least) element, as with the sublattices $(0, 1)$ and R^1 of R^1.

Lemma 2.2.1. *If X' is a sublattice of a lattice X and X'' is a nonempty finite subset of X', then $\sup_X(X'')$ and $\inf_X(X'')$ exist and are contained in X'. Hence, if X is a nonempty finite lattice, then X has a greatest element and a least element.*

Proof. Let $X'' = \{x^1, \ldots, x^k\}$, where k is some positive integer. Let $y^1 = x^1$, so y^1 is in X'. Now suppose that for some integer k' with $1 \leq k' < k$, $y^{k'} = \sup_X(\{x^1, \ldots, x^{k'}\})$ and $y^{k'}$ is in X'. Let $y^{k'+1} = y^{k'} \vee x^{k'+1}$. Because X' is a sublattice of X, $y^{k'+1}$ is in X'. Because $y^{k'} = \sup_X(\{x^1, \ldots, x^{k'}\})$, $y^{k'+1}$ is an upper bound for $\{x^1, \ldots, x^{k'+1}\}$. Let y in X be any upper bound for $\{x^1, \ldots, x^{k'+1}\}$. Because $y^{k'} = \sup_X(\{x^1, \ldots, x^{k'}\})$, it follows that $y^{k'} \preceq y$ and $x^{k'+1} \preceq y$ and so $y^{k'+1} = y^{k'} \vee x^{k'+1} \preceq y$. Consequently, $y^{k'+1} = \sup_X(\{x^1, \ldots, x^{k'+1}\})$. Therefore, by induction, $y^k = \sup_X(\{x^1, \ldots, x^k\}) = \sup_X(X'')$ exists and y^k is in X'. The proof that $\inf_X(X'')$ exists and is contained in X' now follows by considering the dual of X.

The second part of the result follows by letting $X'' = X' = X$. □

If X and T are sets and S is a subset of $X \times T$, then the **section** of S at t in T is $\boldsymbol{S_t} = \{x : x \in X, (x, t) \in S\}$ and the **projection** of S on T is $\boldsymbol{\Pi_T S} = \{t : t \in T, S_t \text{ is nonempty}\}$. Lemma 2.2.2 and Lemma 2.2.3 show that intersections, sections, and projections of sublattices are also sublattices. Corresponding properties hold for convex sets; that is, intersections, sections, and projections of convex sets are also convex. Furthermore, corresponding to part (d) of Example 2.2.3 and part (d) of Example 2.2.5, the direct product of convex sets is convex.

Lemma 2.2.2. *If X is a lattice and X_α is a sublattice of X for each α in a set A, then $\cap_{\alpha \in A} X_\alpha$ is a sublattice of X.*

Proof. Pick any x' and x'' in $\cap_{\alpha \in A} X_\alpha$. For each α in A, x' and x'' are in X_α and X_α is a sublattice of X and so $x' \vee x''$ and $x' \wedge x''$ are in X_α. Therefore, $x' \vee x''$ and $x' \wedge x''$ are in $\cap_{\alpha \in A} X_\alpha$ and so $\cap_{\alpha \in A} X_\alpha$ is a sublattice of X. □

Lemma 2.2.3. *Suppose that X and T are lattices and S is a sublattice of $X \times T$.*
(a) *The section S_t of S at each t in T is a sublattice of X.*
(b) *The projection $\Pi_T S$ of S on T is a sublattice of T.*

Proof. Pick any t in T and x' and x'' in S_t. Because (x', t) and (x'', t) are in S and S is a sublattice of $X \times T$, $(x' \vee x'', t) = (x', t) \vee (x'', t)$ is in S and $(x' \wedge x'', t) = (x', t) \wedge (x'', t)$ is in S. Therefore, $x' \vee x''$ and $x' \wedge x''$ are in S_t, and so S_t is a sublattice of X.

Now pick any t' and t'' in $\Pi_T S$. Then $S_{t'}$ and $S_{t''}$ are nonempty, so there exist some x' in $S_{t'}$ and x'' in $S_{t''}$. Because (x', t') and (x'', t'') are in S and S is a sublattice of $X \times T$, $(x' \vee x'', t' \vee t'') = (x', t') \vee (x'', t'')$ is in S and $(x' \wedge x'', t' \wedge t'') = (x', t') \wedge (x'', t'')$ is in S. Therefore, $t' \vee t''$ and $t' \wedge t''$ are in $\Pi_T S$, and so $\Pi_T S$ is a sublattice of T. □

2.2.2 Sublattice Structure

This subsection proceeds to characterize the structure of sublattices of the direct product of a finite collection of lattices and to use that characterization to characterize those real-valued functions on R^n whose level sets are sublattices of R^n. These results, through Lemma 2.2.7, are largely based on Topkis [1976]. See also Baker and Pixley [1975] and Bergman [1977]. Essential properties characterizing sublattices of the direct product of any finite collection of lattices can be expressed in terms of sublattices of the direct product of two lattices. (The fundamental role of properties on projections of the direct product of only two lattices is also evident in Section 2.6, which characterizes supermodular functions on the direct product of any finite collection of lattices in terms of supermodular functions on the direct product of two lattices.) See Subsection 3.7.4 for characterizations of sublattices of the power set (with the set inclusion ordering relation) of a finite set.

Lemma 2.2.4 characterizes sublattices of the direct product of any finite collection of lattices in terms of sublattices of the direct product of two lattices. Lemma 2.2.5 represents any sublattice of the direct product of two lattices in terms of several easily characterized sublattices. Theorem 2.2.1 uses the results of Lemma 2.2.4 and Lemma 2.2.5 to provide a more refined characterization of the structure of sublattices of the direct product of any finite collection of lattices. The characterizations of sublattice structure are then used in Theorem 2.2.2 to analogously characterize functions whose level sets are sublattices and to exhibit in Example 2.2.7 examples of such functions on R^n.

Suppose that $X_1, \ldots, X_n$ are sets and that elements of $\times_{i=1}^n X_i$ are denoted $x = (x_1, \ldots, x_n)$, where x_i is in X_i for $i = 1, \ldots, n$. If L is a subset of $X_{i'} \times X_{i''}$ for two distinct indices i' and i'' and $S = \{x : x \in \times_{i=1}^n X_i, (x_{i'}, x_{i''}) \in L\}$, then S is a **bivariate** subset of $\times_{i=1}^n X_i$ and L is the **$i'i''$-generator** of S. If $X_1, \ldots, X_n$ are nonempty lattices, S is a bivariate subset of $\times_{i=1}^n X_i$, and L is the $i'i''$-generator of S, then S is a sublattice of $\times_{i=1}^n X_i$ if and only if L is a sublattice of $X_{i'} \times X_{i''}$. Lemma 2.2.4 represents any sublattice of $\times_{i=1}^n X_i$, where $X_1, \ldots, X_n$ are lattices, in terms of $n(n-1)/2$ bivariate sublattices.

Lemma 2.2.4. *If $n \geq 2$ and $X_1, \ldots, X_n$ are lattices, then a set is a sublattice of $\times_{i=1}^n X_i$ if and only if it is the intersection of $n(n-1)/2$ bivariate sublattices of $\times_{i=1}^n X_i$.*

Proof. The sufficiency part is immediate because the intersection of sublattices is a sublattice by Lemma 2.2.2.

Now suppose that S is a nonempty sublattice of $\times_{i=1}^n X_i$. For distinct i' and i'' in $\{1, \ldots, n\}$, let $L_{i',i''}$ be the projection of S on $X_{i'} \times X_{i''}$ and let $S_{i',i''}$ be the bivariate subset of $\times_{i=1}^n X_i$ for which $L_{i',i''}$ is the $i'i''$-generator. Each projection $L_{i',i''}$ is a sublattice of $X_{i'} \times X_{i''}$ by part (b) of Lemma 2.2.3, and so each bivariate set $S_{i',i''}$ is a sublattice of $\times_{i=1}^n X_i$ as in part (d) of Example 2.2.5.

If x is in S, then $(x_{i'}, x_{i''})$ is in $L_{i',i''}$ and so x is in $S_{i',i''}$ for all distinct i' and i''. Therefore, S is a subset of $\cap_{\{i',i'':1 \leq i' \leq n, 1 \leq i'' \leq n, i' \neq i''\}} S_{i',i''}$.

Pick any x in $\cap_{\{i',i'':1 \leq i' \leq n, 1 \leq i'' \leq n, i' \neq i''\}} S_{i',i''}$. For any distinct i' and i'', x is in $S_{i',i''}$ and so $(x_{i'}, x_{i''})$ is in the projection $L_{i',i''}$ of S on $X_{i'} \times X_{i''}$ and there exists some $y^{i',i''}$ in S with $(y_{i'}^{i',i''}, y_{i''}^{i',i''}) = (x_{i'}, x_{i''})$. For each i', let $z^{i'} = \inf\{y^{i',i''} : i'' \neq i'\}$. By Lemma 2.2.1, z^i is in S for each i. Also by Lemma 2.2.1, $\sup\{z^i : i = 1, \ldots, n\}$ is in S. Because $y_{i'}^{i',i''} = x_{i'}$ for all distinct i' and i'', $z_i^i = x_i$ for each i. Because $z_{i''}^{i'} \preceq y_{i''}^{i',i''} = x_{i''}$ for all distinct i' and i'', $z^i \preceq x$ for each i. Therefore, $x = \sup\{z^i : i = 1, \ldots, n\}$ and x is in S. Hence, $\cap_{\{i',i'':1 \leq i' \leq n, 1 \leq i'' \leq n, i' \neq i''\}} S_{i',i''}$ is a subset of S.

The necessity part of this result follows because $S = \cap_{\{i',i'':1 \leq i' \leq n, 1 \leq i'' \leq n, i' \neq i''\}} S_{i',i''}$ and $S_{i',i''} = S_{i'',i'}$ for all distinct i' and i''. □

If X_1 and X_2 are partially ordered sets and S is a subset of $X_1 \times X_2$, then the two **bimonotone hulls** generated by S on $X_1 \times X_2$ are

$$H_1(S) = \cup_{(x_1,x_2) \in S}([x_1, \infty) \times (-\infty, x_2])$$

and

$$H_2(S) = \cup_{(x_1,x_2)\in S}((-\infty, x_1] \times [x_2, \infty)).$$

Part (c) of Lemma 2.2.5 characterizes sublattices of the direct product of two lattices in terms of the two bimonotone hulls and the two projections. Part (a) of Lemma 2.2.5 shows that the bimonotone hulls of a sublattice of the direct product of two lattices are also sublattices. The bimonotone hulls of a subset S of $X_1 \times X_2$ are not generally sublattices if X_1 and X_2 are lattices but not chains and S is not a sublattice of $X_1 \times X_2$. This can be seen from the example with $X_1 = X_2 = R^2$ and $S = \{(0,1,0,1),(1,0,1,0)\}$, because then $(0,1,0,1)$ and $(1,0,1,0)$ are in both bimonotone hulls while $(1,1,1,1) = (0,1,0,1) \vee (1,0,1,0)$ and $(0,0,0,0) = (0,1,0,1) \wedge (1,0,1,0)$ are not in either bimonotone hull. Part (b) of Lemma 2.2.5 gives a surprisingly simple "convex"-like representation for sections of a sublattice of the direct product of two lattices. For the statement and proof of Lemma 2.2.5, note that S_{x_2} denotes the section of S at x_2 in X_2 and that $\Pi_{X_1}S$ and $\Pi_{X_2}S$ denote, respectively, the projections of S on X_1 and on X_2.

Lemma 2.2.5. *Suppose that X_1 and X_2 are lattices and S is a sublattice of $X_1 \times X_2$.*
(a) *The bimonotone hulls of S are sublattices of $X_1 \times X_2$.*
(b) *If x_2 is in X_2 and x_1' and x_1'' are in the section S_{x_2}, then $[x_1', x_1''] \cap \Pi_{X_1}S$ is contained in S_{x_2}. Hence, if x_2 is in X_2 and $\inf_{X_1}(S_{x_2})$ and $\sup_{X_1}(S_{x_2})$ are in S_{x_2}, then $S_{x_2} = [\inf_{X_1}(S_{x_2}), \sup_{X_1}(S_{x_2})] \cap \Pi_{X_1}S$.*
(c) *The sublattice S is the intersection of its two bimonotone hulls and the direct product of its two projections.*

Proof. Pick (x_1', x_2') and (x_1'', x_2'') in the bimonotone hull $H_1(S)$. Then there exist (z_1', z_2') and (z_1'', z_2'') in S with $z_1' \preceq x_1'$, $x_2' \preceq z_2'$, $z_1'' \preceq x_1''$, and $x_2'' \preceq z_2''$. Because S is a sublattice of $X_1 \times X_2$, $(z_1', z_2') \vee (z_1'', z_2'') = (z_1' \vee z_1'', z_2' \vee z_2'')$ and $(z_1', z_2') \wedge (z_1'', z_2'') = (z_1' \wedge z_1'', z_2' \wedge z_2'')$ are in S. Therefore,

$$\begin{aligned}(x_1', x_2') \vee (x_1'', x_2'') &= (x_1' \vee x_1'', x_2' \vee x_2'') \in [z_1' \vee z_1'', \infty) \times (-\infty, z_2' \vee z_2''] \\ &\subseteq H_1(S)\end{aligned}$$

and

$$\begin{aligned}(x_1', x_2') \wedge (x_1'', x_2'') &= (x_1' \wedge x_1'', x_2' \wedge x_2'') \in [z_1' \wedge z_1'', \infty) \times (-\infty, z_2' \wedge z_2''] \\ &\subseteq H_1(S).\end{aligned}$$

Thus, $H_1(S)$ is a sublattice of $X_1 \times X_2$. The proof that $H_2(S)$ is a sublattice of $X_1 \times X_2$ follows similarly, completing the proof of part (a).

Pick any x_2 in X_2, x_1' in S_{x_2}, and x_1'' in S_{x_2}. Then pick z_1 in $[x_1', x_1''] \cap \Pi_{X_1} S$. Because z_1 is in $\Pi_{X_1} S$, there exists z_2 in X_2 with (z_1, z_2) in S. Because $x_1' \preceq z_1$ and S is a sublattice of $X_1 \times X_2$, $(z_1, x_2 \vee z_2) = (x_1', x_2) \vee (z_1, z_2)$ is in S. Because $z_1 \preceq x_1''$ and S is a sublattice of $X_1 \times X_2$, $(z_1, x_2) = (x_1'', x_2) \wedge (z_1, x_2 \vee z_2)$ is in S. Therefore, z_1 is in S_{x_2}, completing the proof of part (b).

Pick any (z_1, z_2) in $H_1(S) \cap H_2(S) \cap (\Pi_{X_1} S \times \Pi_{X_2} S)$. Since (z_1, z_2) is in $H_1(S)$ and in $H_2(S)$, there exist (x_1', x_2') and (x_1'', x_2'') in S such that $x_1' \preceq z_1$, $z_2 \preceq x_2'$, $z_1 \preceq x_1''$, and $x_2'' \preceq z_2$. Because S is a sublattice of $X_1 \times X_2$, $(x_1'', x_2') = (x_1', x_2') \vee (x_1'', x_2'')$ is in S and $(x_1', x_2'') = (x_1', x_2') \wedge (x_1'', x_2'')$ is in S. Thus x_1' is in $S_{x_2'}$ and x_1'' is in $S_{x_2'}$, so $[x_1', x_1''] \cap \Pi_{X_1} S$ is a subset of $S_{x_2'}$ by part (b). Therefore, z_1 is in $S_{x_2'}$ and (z_1, x_2') is in S. Also, x_2' is in the section of S at x_1'' in X_1, and x_2'' is in the section of S at x_1'' in X_1, so $[x_2'', x_2'] \cap \Pi_{X_2} S$ is contained in the section of S at x_1'' in X_1 by part (b) (with the roles of the sets X_1 and X_2 reversed). Therefore, z_2 is contained in the section of S at x_1'' in X_1 and (x_1'', z_2) is in S. Because S is a sublattice of $X_1 \times X_2$, $(z_1, z_2) = (z_1, x_2') \wedge (x_1'', z_2)$ is in S and so $H_1(S) \cap H_2(S) \cap (\Pi_{X_1} S \times \Pi_{X_2} S)$ is a subset of S. The observation that S is a subset of $H_1(S) \cap H_2(S) \cap (\Pi_{X_1} S \times \Pi_{X_2} S)$ completes the proof of part (c). □

Suppose that X_1 and X_2 are partially ordered sets and S is a subset of $X_1 \times X_2$. The subset S is **bimonotone** if either $[x_1, \infty) \times (-\infty, x_2]$ is contained in S for each (x_1, x_2) in S or $(-\infty, x_1] \times [x_2, \infty)$ is contained in S for each (x_1, x_2) in S; that is, if S equals one of its bimonotone hulls. The two bimonotone hulls of S are bimonotone. If X_1 and X_2 are lattices and S is a sublattice of $X_1 \times X_2$, then the two bimonotone hulls of S are sublattices by part (a) of Lemma 2.2.5. Any bimonotone set containing S must also contain at least one of the two bimonotone hulls of S. If X_1 and X_2 are chains, then any bimonotone subset of $X_1 \times X_2$ (and hence each bimonotone hull of any subset of $X_1 \times X_2$) is a sublattice of $X_1 \times X_2$. However, a bimonotone subset of the direct product of two lattices need not be a sublattice, as in the example of the paragraph preceding Lemma 2.2.5 showing that the bimonotone hulls of a subset of the direct product of two lattices need not be a sublattice. If $X_1, \ldots, X_n$ are partially ordered sets, S is a bivariate subset of $\times_{i=1}^n X_i$, a subset L of $X_{i'} \times X_{i''}$ for two distinct indices i' and i'' is the $i'i''$-generator of S, and L is bimonotone, then S is **bimonotone**. Theorem 2.2.1 uses Lemma 2.2.4 and part (a) and part (c) of Lemma 2.2.5 to characterize the sublattices of the direct product of n lattices in terms of $n(n-1)$ bimonotone sublattices and the projections on each of the n lattices.

Theorem 2.2.1. *If $X_1, \ldots, X_n$ are lattices, then a set S is a sublattice of $\times_{i=1}^n X_i$ if and only if it is the intersection of $n(n-1)$ bimonotone sublattices of $\times_{i=1}^n X_i$*

together with the direct product of the projections of S on each X_i where the projection of S on each X_i is a sublattice of X_i.

Proof. The sufficiency part is immediate because the intersection of sublattices is a sublattice by Lemma 2.2.2 and the direct product of sublattices is itself a sublattice as in part (d) of Example 2.2.5.

Suppose that S is a sublattice of $\times_{i=1}^{n} X_i$ and $n \geq 2$. By Lemma 2.2.4, S is the intersection of $n(n-1)/2$ bivariate sublattices of $\times_{i=1}^{n} X_i$. The projection of S on each X_i must equal the intersection of the projections of the $n(n-1)/2$ bivariate sublattices on X_i and each such projection is a sublattice of X_i by part (b) of Lemma 2.2.3. By part (a) and part (c) of Lemma 2.2.5, each bivariate sublattice is the intersection of two bimonotone sublattices of $\times_{i=1}^{n} X_i$ and the direct product of the projections of the bivariate sublattice on each X_i. Therefore, S is the intersection of $n(n-1)$ bimonotone sublattices of $\times_{i=1}^{n} X_i$ together with the direct product of the projections of S on each X_i. □

Sets of feasible decisions are often constructed as the level set of some function (or, equivalently, as the intersection of level sets of some collection of functions), and the present monograph considers structured optimal decision problems for which constraint sets are sublattices. Therefore, it is useful to characterize those functions whose level sets are sublattices in order to facilitate generating and recognizing suitably structured scenarios. If $f(x)$ is a function from a lattice X into a partially ordered set and if each level set of $f(x)$ on X is a sublattice of X, then $f(x)$ is a **sublattice-generating function**.

The preceding results on sublattice structure lead to analogous results, given below, characterizing the structure of sublattice-generating functions. (The two characterizations are equivalent. Indeed, one could instead first directly characterize sublattice-generating functions and then use those results to develop characterizations of sublattice structure.) Generalized indicator functions are used to relate characterizations of sublattices and characterizations of sublattice-generating functions. A generalized indicator function for a subset X' of a lattice X is a sublattice-generating function if and only if X' is a sublattice of X. By this relationship, properties of sublattices can be directly translated into properties of sublattice-generating generalized indicator functions and properties of sublattice-generating functions can be translated into properties of sublattices. Furthermore, part (b) of Lemma 2.2.6 shows that a sublattice-generating function that is bounded above is the pointwise infimum of a collection of sublattice-generating generalized indicator functions, so properties of sublattice-generating generalized indicator functions imply properties of sublattice-generating functions that are bounded above. Following Lemma 2.2.6, the properties developed above on the structure of sublattices are used in Theorem 2.2.2 to give corresponding properties and characterizations of sublattice-generating functions that are bounded above. Lemma 2.2.7 describes separable sublattice-generating functions. Example 2.2.7 exhibits systems of inequality constraints on R^n that determine sublattices of R^n.

Lemma 2.2.6. *Suppose that $f(x)$ is a function from a lattice X into a chain Y.*
(a) *If $f(x)$ is the pointwise infimum of a collection of sublattice-generating functions, then $f(x)$ is a sublattice-generating function.*
(b) *Suppose also that Y has at least two distinct elements. The function $f(x)$ is a sublattice-generating function and is bounded above if and only if it is the pointwise infimum of a collection of sublattice-generating generalized indicator functions.*

Proof. Suppose that $f(x) = \inf\{f_\alpha(x) : \alpha \in A\}$ for each x in X, where $f_\alpha(x)$ is a sublattice-generating function on X for each α in A. Pick any y in Y. Then

$$\begin{aligned}\{x : x \in X, y \preceq f(x)\} &= \{x : x \in X, y \preceq \inf\{f_\alpha(x) : \alpha \in A\}\} \\ &= \{x : x \in X, y \preceq f_\alpha(x) \text{ for each } \alpha \in A\} \\ &= \cap_{\alpha \in A}\{x : x \in X, y \preceq f_\alpha(x)\},\end{aligned}$$

which is a sublattice of X because each $f_\alpha(x)$ is a sublattice-generating function on X and the intersection of sublattices is a sublattice by Lemma 2.2.2. Therefore, $f(x)$ is a sublattice-generating function on X, completing the proof of part (a).

Sufficiency in part (b) follows from part (a) and because any generalized indicator function is bounded above.

Now suppose that $f(x)$ is a sublattice-generating function and is bounded above. Necessity in part (b) is immediate if $f(x)$ is a constant function (using the hypothesis that Y has at least two elements), so suppose that $f(x)$ is not a constant function. If $\sup_Y\{f(x) : x \in X\}$ exists, then set $y' = \sup_Y\{f(x) : x \in X\}$. Otherwise, let y' be any upper bound in Y for $\{f(x) : x \in X\}$. For each y in Y, define

$$X_y = \{x : x \in X, y \prec f(x)\}$$

and

$$g_y(x) = \begin{cases} y' & \text{if } x \in X_y \\ y & \text{if } x \in X \text{ and } x \notin X_y. \end{cases}$$

Let $Y' = \{y : y \in Y \text{ and } X_y \text{ is nonempty}\}$. Because $f(x)$ is a sublattice-generating function on X and Y is a chain, each X_y is a sublattice of X

and so $g_y(x)$ is a sublattice-generating generalized indicator function on X for each y in Y'. Pick any x' in X with $X_{f(x')}$ nonempty. Such x' exists because $f(x)$ is not a constant function and Y is a chain. If y is in Y' and $y \prec f(x')$, then x' is in X_y and $f(x') \preceq y' = g_y(x')$. If y is in Y' and $f(x') \preceq y$, then x' is not in X_y and $f(x') \preceq y = g_y(x')$. Thus, $f(x') \preceq g_y(x')$ for each y in Y'. Furthermore, $f(x')$ is in Y' and x' is not in $X_{f(x')}$, and so $g_{f(x')}(x') = f(x')$. Therefore, $f(x') = \inf\{g_y(x') : y \in Y'\}$ and $f(x)$ is the pointwise infimum of a collection of sublattice-generating generalized indicator functions on X. $\square$

Suppose that $X_1, \ldots, X_n$ are sets, elements of $\times_{i=1}^n X_i$ are denoted $x = (x_1, \ldots, x_n)$ where x_i is in X_i for $i = 1, \ldots, n$, and $f(x)$ is a function on $\times_{i=1}^n X_i$. If there is some i' such that $f(x)$ does not depend on x_i for any $i \neq i'$, then $f(x)$ is **univariate**. If there are some distinct i' and i'' such that $f(x)$ does not depend on x_i for any i with $i \neq i'$ and $i \neq i''$, then $f(x)$ is **bivariate**. A generalized indicator function of a bivariate set is bivariate. If each X_i is a partially ordered set, the range of $f(x)$ on $\times_{i=1}^n X_i$ is a partially ordered set, there are distinct i' and i'' such that $f(x)$ does not depend on x_i for any i with $i \neq i'$ and $i \neq i''$, $f(x)$ is increasing in $x_{i'}$, and $f(x)$ is decreasing in $x_{i''}$, then $f(x)$ is **bimonotone**. A generalized indicator function of a bimonotone set is bimonotone. Each level set of a bimonotone function is bimonotone. Definitions of **univariate**, **bivariate**, and **bimonotone** apply for functions on R^n, where the domain is considered as the direct product $\times_{i=1}^n X_i$ with $X_i = R^1$ for each i; likewise, for functions on a product set that is contained in R^n. Theorem 2.2.2 uses Lemma 2.2.6 and the preceding characterizations of sublattice structure to characterize sublattice-generating functions. In particular, any sublattice-generating function that is bounded above on the direct product of n chains is characterized in terms of $n(n-1)$ bimonotone (sublattice-generating) functions and n univariate (sublattice-generating) functions.

Theorem 2.2.2. *Suppose that $n \geq 2$, $X_1, \ldots, X_n$ are lattices, and $f(x)$ is a function from $\times_{i=1}^n X_i$ into a chain in which each nonempty subset that is bounded below has a greatest lower bound, where $x = (x_1, \ldots, x_n)$ and x_i is in X_i for $i = 1, \ldots, n$.*
(a) *The function $f(x)$ is a sublattice-generating function and is bounded above if and only if it is the pointwise infimum of $n(n-1)/2$ bivariate sublattice-generating functions that are bounded above.*
(b) *The function $f(x)$ is a sublattice-generating function and is bounded above if and only if it is the pointwise infimum of $n(n-1)$ bimonotone sublattice-generating functions that are bounded above and n univariate sublattice-generating functions that are bounded above.*

Now suppose that, in addition, each X_i is a chain.
(c) *If $f(x)$ is univariate, then $f(x)$ is a sublattice-generating function.*
(d) *If $f(x)$ is bimonotone, then $f(x)$ is a sublattice-generating function.*
(e) *If $f(x)$ is the pointwise infimum of $n(n-1)$ bimonotone functions and n univariate functions, then $f(x)$ is a sublattice-generating function.*
(f) *If $f(x)$ is a sublattice-generating function and is bounded above, then it is the pointwise infimum of $n(n-1)$ bimonotone functions that are bounded above and n univariate functions that are bounded above.*

Proof. Sufficiency in part (a) and part (b) follows from part (a) of Lemma 2.2.6.

Suppose that $f(x)$ is a sublattice-generating function, is bounded above, and is not a constant function. By part (b) of Lemma 2.2.6, $f(x)$ is the pointwise infimum of a collection of sublattice-generating generalized indicator functions. By Lemma 2.2.4, any sublattice-generating generalized indicator function is the pointwise infimum of at most $n(n-1)/2$ bivariate sublattice-generating generalized indicator functions. Therefore, $f(x)$ is the pointwise infimum of a collection of bivariate sublattice-generating generalized indicator functions and hence, using part (a) of Lemma 2.2.6, $f(x)$ is the pointwise infimum of $n(n-1)/2$ bivariate sublattice-generating functions that are bounded above. This establishes necessity in part (a).

Again, suppose that $f(x)$ is a sublattice-generating function, is bounded above, and is not a constant function. By Theorem 2.2.1, a sublattice-generating generalized indicator function is the pointwise infimum of at most $n(n-1)$ bimonotone sublattice-generating generalized indicator functions and at most n univariate sublattice-generating generalized indicator functions. Therefore, using part (b) of Lemma 2.2.6, $f(x)$ is the pointwise infimum of a collection of bimonotone sublattice-generating generalized indicator functions and univariate sublattice-generating generalized indicator functions. Hence, using part (a) of Lemma 2.2.6, $f(x)$ is the pointwise infimum of $n(n-1)$ bimonotone sublattice-generating functions and n univariate sublattice-generating functions. This establishes necessity in part (b).

Suppose that each X_i is a chain and $f(x)$ is univariate on $\times_{i=1}^n X_i$. Each level set of $f(x)$ is the direct product of chains and so is a sublattice of $\times_{i=1}^n X_i$. Therefore, $f(x)$ is a sublattice-generating function, establishing part (c).

Suppose that each X_i is a chain and $f(x)$ is bimonotone on $\times_{i=1}^n X_i$. Each level set of $f(x)$ is a bimonotone subset of $\times_{i=1}^n X_i$. Because each bimonotone subset of the direct product of chains is a sublattice, each level set of $f(x)$ is a sublattice of $\times_{i=1}^n X_i$. Therefore, $f(x)$ is a sublattice-generating function, establishing part (d).

Part (e) follows from part (c) and part (d) and from part (a) of Lemma 2.2.6.

Part (f) is a consequence of part (b), part (c), and part (d). □

A real-valued function $f(x)$ on a product set $\times_{i=1}^n X_i$ is **separable** if $f(x) = \sum_{i=1}^n f_i(x_i)$ for all $x = (x_1, \ldots, x_n)$ with x_i in X_i for $i = 1, \ldots, n$. Lemma 2.2.7 shows that any separable sublattice-generating function on the direct product of chains must be either univariate or bimonotone.

Lemma 2.2.7. *A real-valued separable sublattice-generating function on the direct product of a finite collection of chains is either univariate or bimonotone.*

Proof. Let X_i be a chain for $i = 1, \ldots, n$ and let $f(x) = \sum_{i=1}^n f_i(x_i)$ be a real-valued separable function on $\times_{i=1}^n X_i$. Suppose that $f(x)$ is neither univariate nor bimonotone. Then there exist distinct i' and i'', $x'_{i'}$ in $X_{i'}$, $x''_{i'}$ in $X_{i'}$, $x'_{i''}$ in $X_{i''}$, and $x''_{i''}$ in $X_{i''}$ such that $x'_{i'} \prec x''_{i'}$, $x''_{i''} \prec x'_{i''}$, and either $f_{i'}(x'_{i'}) < f_{i'}(x''_{i'})$ and $f_{i''}(x''_{i''}) < f_{i''}(x'_{i''})$ or $f_{i'}(x''_{i'}) < f_{i'}(x'_{i'})$ and $f_{i''}(x'_{i''}) < f_{i''}(x''_{i''})$. Consider the former case. (The proof for the latter case is similar.) For each i with $i \neq i'$ and $i \neq i''$, pick any x'_i in X_i. Let $x''_i = x'_i$ for each i with $i \neq i'$ and $i \neq i''$, $x' = (x'_1, \ldots, x'_n)$, and $x'' = (x''_1, \ldots, x''_n)$. Then $f(x') < f(x' \vee x'')$, $f(x'') < f(x' \vee x'')$, $f(x' \wedge x'') < f(x')$, and $f(x' \wedge x'') < f(x'')$. Let $y = \min\{f(x'), f(x'')\}$. Then the level set $\{x : x \in \times_{i=1}^n X_i, y \leq f(x)\}$ is not a sublattice of $\times_{i=1}^n X_i$ because it contains x' and x'' but not $x' \wedge x''$. Therefore, $f(x)$ is not a sublattice-generating function. □

Based on the characterization of Theorem 2.2.2 together with Lemma 2.2.2, Example 2.2.7 exhibits systems of inequality constraints on R^n that determine sublattices of R^n.

Example 2.2.7. The set of feasible solutions for each of the following systems of inequality constraints on R^n is a sublattice of R^n.

(a) The set

$$\{x : x \in R^n, 0 \leq g_i(x_i) \text{ for } i = 1, \ldots, n, \text{ and } 0 \leq f_{i',i''}(x_{i'}, x_{i''}) \text{ for all distinct integers } i' \text{ and } i'' \text{ with } 1 \leq i' \leq n \text{ and } 1 \leq i'' \leq n\}$$

is a sublattice of R^n if $g_i(x_i)$ is a real-valued function on R^1 for $i = 1, \ldots, n$ and if $f_{i',i''}(x_{i'}, x_{i''})$ is a real-valued bimonotone function on R^2 for all distinct i' and i'' in $\{1, \ldots, n\}$. Conversely, any real-valued sublattice-generating function that is bounded above on R^n has a representation as the pointwise infimum of n^2 such functions. That is, in R^n the real-valued sublattice-generating functions that are bounded above are precisely those functions that can be represented as a system of n^2 inequality constraints, where n of the inequality constraints ($g_i(x_i)$ for $i = 1, \ldots, n$) are bounded above and are functions of only

one real component and $n(n-1)$ of the inequality constraints ($f_{i',i''}(x_{i'}, x_{i''})$ for all distinct i' and i'') are bounded above and are functions of only two real components and are increasing in one of these components and decreasing in the other component.
(b) Suppose that a^j is an n-vector for $j = 1, \ldots, m$. Let $b = (b_1, \ldots, b_m)$ be an arbitrary m-vector. Consider the set of solutions to the system of linear inequalities

$$S_b = \{x : x \in R^n, b_j \leq a^j \cdot x \text{ for } j = 1, \ldots m\}.$$

Then S_b is a sublattice of R^n for each b in R^m if and only if each n-vector a^j with more than one nonzero component has exactly two nonzero components and these are of opposite signs. This system includes as special cases the dual of the transportation problem and the dual of the generalized transportation problem (Ahuja, Magnanti, and Orlin [1993]). See Section 3.4.
(c) A firm must select some subset of n activities, denoted $i = 1, \ldots, n$, in which to engage. The firm's selection of activities is indicated by an n-vector $x = (x_1, \ldots, x_n)$, where $x_i = 1$ if the firm decides to engage in activity i and $x_i = 0$ otherwise. Some of the activities depend on the availability of certain other activities, so for each activity i there is a subset A_i of the other activities that must be selected if activity i is to be selected. This constraint can be represented as $0 \leq x_{i''} - x_{i'}$ for each activity i' and each activity i'' in $A_{i'}$; that is, $x_{i'}$ can equal 1 only if $x_{i''} = 1$ for each activity i'' in $A_{i'}$. Then the collection of all feasible sets of activities in which the firm may choose to engage is represented by

$$\{x : x \in R^n, x_i \text{ equals 0 or 1 for each } i, \text{ and } 0 \leq x_{i''} - x_{i'} \text{ for all } i' \text{ and } i'' \text{ with } i'' \in A_{i'}\},$$

which is a sublattice of R^n. See Subsection 3.7.3.
(d) An acyclic network consists of a collection of nodes $i = 1, \ldots, n$ and a set of directed edges joining ordered pairs of distinct nodes such that an edge (i', i'') exists from node i' to node i'' if and only if $i' < i''$. Then the set of edges

$$\{(i', i'') : 1 \leq i' < i'' \leq n, i' \text{ and } i'' \text{ integer}\}$$

is a sublattice of R^2. See Subsection 3.5.1.

For a sublattice of the direct product of a finite collection of chains, Theorem 2.2.3 gives structural properties of minimal sets of indices whose components may strictly increase within the sublattice above their values for any particular element of the sublattice. This result is useful in developing properties of functions whose domain is a sublattice of the direct product of finitely many chains. (See Theorem 2.6.3, Theorem 2.7.4, and Theorem 2.8.12.) Suppose that $X_1, \ldots, X_n$ are chains and S is a sublattice of $\times_{i=1}^n X_i$. A subset I of $\{1, \ldots, n\}$ is an **increasable** set of indices for x' in S if there exists x'' in S with $x' \preceq x''$ and with $x'_i \prec x''_i$ if and only if i is in I. In this case, x'' **corresponds to** the increasable set of indices I for x' in S. The collection of all increasable sets of indices for x' in S is closed under union and intersection; that is, it is a sublattice of $\{1, \ldots, n\}$ with ordering relation $\subseteq$. An increasable set of indices I is an **increasable cover** for x' in S if I is a cover for the empty set in the collection of all increasable sets of indices for x'; that is, if I is a nonempty increasable set for x' and there does not exist an increasable set of indices I' for x' with $\emptyset \subset I' \subset I$. (An n-vector that corresponds to an increasable cover for x' in S need not cover x' in S. There may be many n-vectors that correspond to any particular increasable set of indices or increasable cover for x' in S.) If $S = \times_{i=1}^n X_i$, then any increasable cover for x' in S consists of exactly one index element and any n-vector that corresponds to an increasable cover differs from x' in exactly one component. In particular, if $S = R^n$ and x' is in S, then any subset of $\{1, \ldots, n\}$ is an increasable set for x' in S, the sets $\{1\}, \ldots, \{n\}$ are the increasable covers for x' in S, and x'' corresponds to an increasable cover if and only if $x'' = x' + \epsilon u^i$ for some $\epsilon > 0$ and any index i. If S is bivariate, then at most one increasable cover for x' in S consists of exactly two index elements and each other increasable cover for x' consists of exactly one index element. For example, if $S = \{x : x \in R^n, x_1 = x_2\}$ where $n \geq 2$ and x' is in S, then the increasable sets for x' in S are those subsets of $\{1, \ldots, n\}$ that either are disjoint from $\{1, 2\}$ or contain $\{1, 2\}$, the sets $\{1, 2\}, \{3\}, \ldots, \{n\}$ are the increasable covers for x' in S, and x'' corresponds to an increasable cover if and only if either $x'' = x' + \epsilon u^i$ for some $\epsilon > 0$ and some index $i \geq 3$ or $x'' = x' + \epsilon u^1 + \epsilon u^2$ for some $\epsilon > 0$. The increasable sets and the increasable covers may differ for different elements of a sublattice as can be seen with $S = \{x : x \in R^2, x_1 \leq 0, x_2 \leq 0\} \cup \{x : x \in R^2, x_1 = x_2 > 0\}$, where $\{1, 2\}$ is the only nonempty increasable set (and hence the only increasable cover) for x' in S with $x'_1 = x'_2 \geq 0$, $\{1\}$ is the only increasable cover for x' in S with $x'_1 < 0$ and $x'_2 = 0$, $\{2\}$ is the only increasable cover for x' in S with $x'_1 = 0$ and $x'_2 < 0$, and $\{1\}$ and $\{2\}$ are the increasable covers for x' in S with $x'_1 < 0$ and $x'_2 < 0$.

Theorem 2.2.3. *Suppose that $X_1, \ldots, X_n$ are chains and S is a sublattice of $\times_{i=1}^n X_i$.*
(a) *The distinct increasable covers for x' in S are disjoint.*
(b) *If x' and x'' in S correspond to distinct increasable covers for $x' \wedge x''$ in S, then $S \cap (\times_{i=1}^n \{x'_i, x''_i\}) = \{x', x'', x' \vee x'', x' \wedge x''\}$.*

(c) *If* x' *and* x'' *in* S *are unordered and* $S \cap (\times_{i=1}^n \{x'_i, x''_i\}) \neq \{x', x'', x' \vee x'', x' \wedge x''\}$, *then there exist* z' *and* z'' *in* $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$ *such that* $S \cap (\times_{i=1}^n \{z'_i, z''_i\}) = \{z', z'', z' \vee z'', z' \wedge z''\}$, z' *and* z'' *correspond to distinct increasable covers for* $x' \wedge x''$ *in* $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$, $z' \preceq x'$, $z'' \preceq x''$, $x' \prec x' \vee z''$, *and* $x'' \prec x'' \vee z'$.

Proof. Let I' and I'' be distinct increasable covers for x' in S. If z' corresponds to I' and z'' corresponds to I'', then $z' \wedge z''$ is in S with $x' \preceq z' \wedge z''$ and with $x'_i \prec z'_i \wedge z''_i$ if and only if i is in $I' \cap I''$. Thus, $I' \cap I''$ is an increasable set of indices for x' in S. Suppose that I' and I'' are not disjoint, so $I' \cap I''$ is nonempty. Because I' and I'' are distinct either $\emptyset \subset I' \cap I'' \subset I'$ or $\emptyset \subset I' \cap I'' \subset I''$, which contradicts I' and I'' being increasable covers for x' in S. Therefore, I' and I'' are disjoint, establishing part (a).

Suppose that x' and x'' in S correspond to the distinct increasable covers I' and I'' for $x' \wedge x''$ in S. Because S is a sublattice of $\times_{i=1}^n X_i$, $\{x', x'', x' \vee x'', x' \wedge x''\}$ is a subset of $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$. Pick any x in $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$. Let $I = \{i : i = 1, \ldots, n, x'_i \wedge x''_i \prec x_i\}$. Then I is a subset of $I' \cup I''$. By part (a), $I' \cap I'' = \emptyset$. Because S is a sublattice of $\times_{i=1}^n X_i$, $x \wedge x'$ and $x \wedge x''$ are in S and correspond, respectively, to the increasable sets $I \cap I'$ and $I \cap I''$ for $x' \wedge x''$ in S. Either $\emptyset \subset I \cap I' \subset I'$ or $\emptyset \subset I \cap I'' \subset I''$ would contradict I' and I'' being increasable covers. Thus either $I \cap I' = \emptyset$ or I' is a subset of I holds and either $I \cap I'' = \emptyset$ or I'' is a subset of I holds. If $I \cap I' = \emptyset$ and $I \cap I'' = \emptyset$, then I being a subset of $I' \cup I''$ implies that $I = \emptyset$ and so $x = x' \wedge x''$. If I' is a subset of I and I'' is a subset of I, then $I' \cup I''$ is a subset of I so I being a subset of $I' \cup I''$ implies that $I = I' \cup I''$ and $x = x' \vee x''$. If I' is a subset of I and $I \cap I'' = \emptyset$, then I being a subset of $I' \cup I''$ implies that $I = I'$ and so $x = x'$. Likewise, if I'' is a subset of I and $I \cap I' = \emptyset$, then I being a subset of $I' \cup I''$ implies that $I = I''$ and so $x = x''$. Thus, x is in $\{x', x'', x' \vee x'', x' \wedge x''\}$, so $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$ is a subset of $\{x', x'', x' \vee x'', x' \wedge x''\}$ and part (b) holds.

Suppose that x' and x'' in S are unordered and $S \cap (\times_{i=1}^n \{x'_i, x''_i\}) \neq \{x', x'', x' \vee x'', x' \wedge x''\}$. Let I' and I'' be increasable covers for $x' \wedge x''$ in $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$ with I' being a subset of $\{i : x'_i \wedge x''_i \prec x'_i\}$ and I'' being a subset of $\{i : x'_i \wedge x''_i \prec x''_i\}$. Thus $I' \cap I'' = \emptyset$ so I' and I'' are disjoint increasable covers for $x' \wedge x''$ in $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$. Let z' and z'' in $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$ correspond, respectively, to the distinct increasable covers I' and I'' for $x' \wedge x''$ in $S \cap (\times_{i=1}^n \{x'_i, x''_i\})$. Because I' is a subset of $\{i : x'_i \wedge x''_i \prec x'_i\}$ and I'' is a subset of $\{i : x'_i \wedge x''_i \prec x''_i\}$, $z' \preceq x'$ and $z'' \preceq x''$. If $z'' \preceq x'$ then $z'' \preceq x' \wedge x''$ which is a contradiction, so $x' \prec x' \vee z''$. If $z' \preceq x''$ then $z' \preceq x' \wedge x''$ which is a contradiction, so $x'' \prec x'' \vee z'$. The remainder of the proof of part (c) now follows from part (b). $\square$

2.3 Completeness and Topological Properties

This section gives compactness conditions that characterize those lattices and those sublattices of a given lattice for which each nonempty subset has a supremum and an infimum; that is, complete lattices and subcomplete sublattices. A consequence is the identification of conditions implying that a lattice or a sublattice has a greatest element and a least element.

A lattice in which each nonempty subset has a supremum and an infimum is **complete**. By Lemma 2.2.1, any finite lattice is complete. A nonempty complete lattice has a greatest element and a least element. If X' is a sublattice of a lattice X and if, for each nonempty subset X'' of X', $\sup_X(X'')$ and $\inf_X(X'')$ exist and are contained in X', then X' is a **subcomplete** sublattice of X. By Lemma 2.2.1, any finite sublattice of a lattice is subcomplete. Hence, any sublattice of a finite lattice is subcomplete. Each closed interval in a complete lattice X is a subcomplete sublattice of X, and the supremum and infimum with respect to the closed interval of any subset of the closed interval are the same as the supremum and infimum with respect to X of that same subset. (See part (e) of Example 2.2.5.) If X is a lattice and X' is a subcomplete sublattice of X, then $\sup_X(X'') = \sup_{X'}(X'')$ and $\inf_X(X'') = \inf_{X'}(X'')$ for each nonempty subset X'' of X', X' itself is a complete lattice, and X' has a greatest element and a least element if X' is nonempty. If X and X' are complete lattices with the same ordering relation and with X' being a subset of X, then X' need not be a subcomplete sublattice of X as the example with $X = [0, 2]$, $X' = [0, 1) \cup \{2\}$, and $X'' = [0, 1)$ shows (since $\sup_X(X'') = 1$ is not in X').

The **interval topology** on a partially ordered set X is that topology for which each closed set is either X or the empty set or can be represented as the intersection of sets that are finite unions of closed intervals in X. (See part (e) of Example 2.2.5.) Frink [1942] introduces the interval topology and shows that a complete lattice is compact in its interval topology. Conversely, Birkhoff [1967] shows that a lattice that is compact in its interval topology is complete. Unfortunately, completeness and the interval topology are not the most relevant and convenient notions when one is interested in properties of a set relative to its being a subset of some larger set. In particular, the join, meet, supremum, and infimum have convenient forms in a sublattice of R^n that do not generally hold in a lattice contained in R^n. See Example 2.2.2 and Example 2.2.4. The set $X = [0, 1) \cup \{2\}$ in R^1 is complete and is compact in its interval topology, but these properties are not the most relevant ones when X is viewed as a subset of R^1. The collection of closed sets in the interval topology on a subset X of R^n is a subset of the collection of sets that are closed on X relative to the usual topology on R^n (that is, the intersection of X with each closed set in R^n). Furthermore, the former collection of closed sets may be a strict subset of the latter for certain subsets X of R^n. For example, if $X = [0, 1) \cup \{2\}$, then the set $[0, 1)$ is closed in X relative to the usual topology on R^1 but is not closed in the interval topology on X. (Except for the discussion in the present paragraph and in the final paragraph of this section, this monograph assumes throughout the usual topology on R^n.) Theorem 2.3.1 provides a more useful and natural result for sublattices of R^n by showing that subcompleteness is equivalent to compactness in the usual topology. Versions of Theorem 2.3.1 in more general topological spaces are noted in the final paragraph of this section. For a set X in R^n, the **closure** of X is the intersection of all closed sets containing X and is denoted $\boldsymbol{cl(X)}$.

Theorem 2.3.1. *A sublattice of R^n is subcomplete if and only if it is compact.*

Proof. Suppose that X is a compact sublattice of R^n. Pick any nonempty subset X' of X. Because X is compact, $cl(X')$ is a nonempty compact subset of X. For $i = 1, \ldots, n$, $\text{argmax}_{x \in cl(X')} x_i$ is nonempty. Pick any x^i in $\text{argmax}_{x \in cl(X')} x_i$. Let $x' = \sup_{R^n}\{x^1, \ldots, x^n\} = (x^1_1, \ldots, x^n_n)$. By Lemma 2.2.1, x' is in X. Because $x_i \le x^i_i = x'_i$ for $i = 1, \ldots, n$ and each x in $cl(X')$, x' is an upper bound for $cl(X')$ and hence for X'. Let x'' be any upper bound for X'. Because x'' is an upper bound for X', x'' must be an upper bound for $cl(X')$ and so $x^i \le x''$ for $i = 1, \ldots, n$ and $x' \le x''$. Therefore, $x' = \sup_{R^n}(X')$. The proof that $\inf_{R^n}(X')$ exists and is in X now follows by considering the dual of R^n. Thus, X is subcomplete.

Now suppose that X is a nonempty subcomplete sublattice of R^n. Because X is subcomplete, X has a greatest element and a least element and so X is bounded. Pick any x' in $cl(X)$, so there is a sequence $\{x^k : k = 1, 2, \ldots\}$ contained in X with $\lim_{k\to\infty} x^k = x'$. Because X is subcomplete, $z^k = \sup_{R^n}\{x^{k'} : k' = k, k+1, \ldots\}$ exists and is in X for $k = 1, 2, \ldots$. Because X is subcomplete, $x'' = \inf_{R^n}\{z^k : k = 1, 2, \ldots\}$ exists and is in X. Because $z^1 \ge z^2 \ge \ldots$ and $\lim_{k\to\infty} x^k = x'$, $x' = \lim_{k\to\infty} x^k = \lim_{k\to\infty} z^k = x''$ is in X. Thus, X is closed. Because X is closed and bounded, X is compact. □

By Theorem 2.3.1, if X is a compact (or, equivalently, subcomplete) sublattice of R^n and X' is a nonempty subset of X, then $\sup_{R^n}(X') = \sup_X(X')$ and $\inf_{R^n}(X') = \inf_X(X')$ and so one can unambiguously omit the underlying set (X or R^n) and write $\sup(X')$ and $\inf(X')$.

One consequence of Theorem 2.3.1, stated in Corollary 2.3.2, is that any nonempty compact sublattice of R^n has a greatest element and a least element. Corollary 2.3.1 gives a more refined result for a nonempty closed sublattice of R^n that is either bounded above or bounded below.

Corollary 2.3.1. *A nonempty closed sublattice of R^n that is bounded above (below) has a greatest (least) element.*

Proof. Let X be a nonempty closed sublattice of R^n that is bounded above. Pick any x' in X. Let x'' be an upper bound for X in R^n. By Lemma 2.2.2 and part (e) of Example 2.2.5, $X \cap [x', x'']$ is a nonempty compact sublattice of R^n. By Theorem 2.3.1, $X \cap [x', x'']$ is subcomplete and thus has a greatest element x'''. If x is in X, then $x \vee x'$ is in $X \cap [x', x'']$ and so $x \leq x \vee x' \leq x'''$. Therefore, x''' is the greatest element of X. □

Corollary 2.3.2. *A nonempty compact sublattice of R^n has a greatest element and a least element.*

Results given in this section are used directly or indirectly in subsequent results in this chapter (e.g., Theorem 2.4.3, part (b) of Corollary 2.7.1, and part (a) of Theorem 2.8.3) and in applications in subsequent chapters, as well as being relevant for applications of the fixed point results of Section 2.5.

Versions of Theorem 2.3.1 hold in more general topological spaces. If X is a topological space and X' is a subset of X, then the **relative topology** on X' has as its closed sets the intersection of X' with each closed set in the topological space X. If X is a lattice with the interval topology and X' is a sublattice of X, then X' is subcomplete if and only if X' is compact in the relative topology (Topkis [1977]). If X_α is a topological space for each α in a set A, then the **product topology** on the product set $\times_{\alpha \in A} X_\alpha$ is defined to be that topology for which each closed set can be represented as the intersection of sets that are finite unions of sets that are the direct product (over all α in A) of sets that are closed in each topological space X_α. If X_α is a lattice with a greatest element and a least element and with the interval topology for each α in A, then the interval topology on $\times_{\alpha \in A} X_\alpha$ has the same closed sets as the product topology on $\times_{\alpha \in A} X_\alpha$ (Frink [1942]). This statement need not be true if each X_α does not have a greatest element and a least element (although even then any set closed in the interval topology on $\times_{\alpha \in A} X_\alpha$ is closed in the product topology on $\times_{\alpha \in A} X_\alpha$), as can be seen from the example with $R^2 = R^1 \times R^1$ where each R^1 has the usual ordering relation $\leq$ and the interval topology (which is the usual topology on R^1) since the set $\{x : x_1 \geq 0\}$ is closed in the product topology on R^2 but is not closed in the interval topology on R^2. If X_α is a lattice with the interval topology for each α in A and X' is a sublattice of $\times_{\alpha \in A} X_\alpha$, then X' is subcomplete if and only if X' is compact in the relative topology of the product topology on $\times_{\alpha \in A} X_\alpha$ (Topkis [1977]). The various equivalences between completeness or subcompleteness and compactness in certain topological spaces can be used to extend subsequent results in this chapter (e.g., the last part of Theorem 2.4.3, part (b) of Corollary 2.7.1, and part (a) of Theorem 2.8.3) as well as some applications (e.g., the noncooperative game model of Chapter 4) from R^n to other topological spaces.

2.4 Induced Set Ordering

This section provides properties and characterizations of a natural ordering relation, the induced set ordering, on the nonempty sublattices of a given lattice.

Suppose that X is a lattice with ordering relation $\preceq$. The **induced set ordering $\sqsubseteq$**, so-called because it is induced by the ordering relation $\preceq$ on X, is defined on the collection of nonempty members of the power set $\mathcal{P}(X) \setminus \{\emptyset\}$ such that $X' \sqsubseteq X''$ in $\mathcal{P}(X) \setminus \{\emptyset\}$ if x' in X' and x'' in X'' imply that $x' \wedge x''$ is in X' and $x' \vee x''$ is in X''. Subsets of X that are singletons are ordered by $\sqsubseteq$ if and only if their respective elements are correspondingly ordered by $\preceq$; that is, for x' and x'' in X, $\{x'\} \sqsubseteq \{x''\}$ if and only if $x' \preceq x''$.

The induced set ordering is used almost exclusively in this monograph to order sets of feasible solutions and sets of optimal solutions from parameterized collections of optimization problems. As seen in this section, the induced set ordering has a number of useful and desirable properties. As seen in Section 2.8, the induced set ordering has a key role in parameterized optimization problems involving supermodular functions. (See also Section 2.5.) Other set orderings might also be considered. One is the weak induced set ordering used in Theorem 2.4.7 to characterize the induced set ordering. Another ordering would order sets if ordered selections from the sets exist. This ordering is weaker than the induced set ordering. (See Lemma 2.4.2, Theorem 2.4.3, and Theorem 2.4.4.) And another ordering would order sets if all selections from the sets are ordered. (See Theorem 2.8.4.) This ordering is stronger than the induced set ordering. Shannon [1995] studies the latter two orderings and their role in parameterized optimization problems.

Lemma 2.4.1 leads to Theorem 2.4.1, both from Topkis [1978], showing that the collection $\mathcal{L}(X)$ of all nonempty sublattices of a lattice X is the greatest subset (according to the set inclusion ordering) of $\mathcal{P}(X) \setminus \{\emptyset\}$ on which $\sqsubseteq$ is a partial ordering.

Lemma 2.4.1. *If X is a lattice, then the binary relation $\sqsubseteq$ is antisymmetric and transitive on $\mathcal{P}(X) \setminus \{\emptyset\}$.*

Proof. Pick any X' and X'' in $\mathcal{P}(X) \setminus \{\emptyset\}$ for which $X' \sqsubseteq X''$ and $X'' \sqsubseteq X'$. Now pick any x' in X' and x'' in X''. Because $X' \sqsubseteq X''$, $x' \wedge x''$ is in X' and $x' \vee x''$ is in X''. But then because $X'' \sqsubseteq X'$, $x'' = x'' \vee (x' \wedge x'')$ is in X'

and $x' = (x' \vee x'') \wedge x'$ is in X''. Thus $X' = X''$ and so $\sqsubseteq$ is antisymmetric on $\mathcal{P}(X) \setminus \{\emptyset\}$.

Pick any X', X'', and X''' in $\mathcal{P}(X) \setminus \{\emptyset\}$ such that $X' \sqsubseteq X''$ and $X'' \sqsubseteq X'''$. Now pick any x' in X', x'' in X'', and x''' in X'''. Since $X' \sqsubseteq X''$ and $X'' \sqsubseteq X'''$, $x' \vee x''$ is in X'' and $x'' \wedge x'''$ is in X''. Then $X' \sqsubseteq X''$ implies that $x' \vee (x'' \wedge x''')$ is in X'' and so $x' \vee x''' = x' \vee ((x'' \wedge x''') \vee x''') = (x' \vee (x'' \wedge x''')) \vee x'''$ is in X''' because $X'' \sqsubseteq X'''$. Similarly, $X'' \sqsubseteq X'''$ implies that $(x' \vee x'') \wedge x'''$ is in X'' and so $x' \wedge x''' = (x' \wedge (x' \vee x'')) \wedge x''' = x' \wedge ((x' \vee x'') \wedge x''')$ is in X' because $X' \sqsubseteq X''$. Therefore, $X' \sqsubseteq X'''$ and so $\sqsubseteq$ is transitive on $\mathcal{P}(X) \setminus \{\emptyset\}$.
□

Neither the antisymmetric property nor the transitive property of Lemma 2.4.1 generally holds if one includes the empty set and considers the binary relation $\sqsubseteq$ on $\mathcal{P}(X)$ rather than just on $\mathcal{P}(X) \setminus \{\emptyset\}$.

For a lattice X, it follows directly from the definition of a sublattice that X' is in $\mathcal{L}(X)$ if and only if X' is in $\mathcal{P}(X) \setminus \{\emptyset\}$ and $X' \sqsubseteq X'$; that is, $\mathcal{L}(X)$ is the greatest subset (according to the set inclusion ordering on $\mathcal{P}(\mathcal{P}(X) \setminus \{\emptyset\})$) of $\mathcal{P}(X) \setminus \{\emptyset\}$ on which $\sqsubseteq$ is reflexive. This observation together with Lemma 2.4.1 yields the following result.

Theorem 2.4.1. *If X is a lattice, then $\mathcal{L}(X)$ is a partially ordered set with the ordering relation $\sqsubseteq$. Furthermore, any subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ that is a partially ordered set with the ordering relation $\sqsubseteq$ is a subset of $\mathcal{L}(X)$.*

Unless explicitly stated otherwise in a particular context, this monograph assumes throughout that whenever any subset of $\mathcal{L}(X)$ is considered as a partially ordered set for any lattice X then the ordering relation on $\mathcal{L}(X)$ is the induced set ordering $\sqsubseteq$. A function whose range is included in the collection of all subsets of some set is a **correspondence**. A correspondence S_t is **increasing** (**decreasing**) in t on T if the domain T is a partially ordered set, the range $\{S_t : t \in T\}$ is in $\mathcal{L}(X)$ where X is a lattice and $\mathcal{L}(X)$ is a partially ordered set with the ordering relation $\sqsubseteq$, and S_t is an increasing (decreasing) correspondence from T into $\mathcal{L}(X)$ (so $t' \preceq t''$ in T implies $S_{t'} \sqsubseteq S_{t''}$ ($S_{t''} \sqsubseteq S_{t'}$) in $\mathcal{L}(X)$). In stating that a correspondence S_t is increasing (decreasing), it is typically left implicit herein that each S_t is a nonempty sublattice of X and that $\sqsubseteq$ is the ordering relation on the sets S_t in $\mathcal{L}(X)$. A collection of optimization problems as in (2.1.1), maximize $f(x, t)$ subject to x in S_t, with each problem determined by a parameter t in a partially ordered set T and with each constraint set S_t contained in a lattice X has **increasing (decreasing) optimal solutions** if the correspondence $\text{argmax}_{x \in S_t} f(x, t)$ is increasing (decreasing) as a function of t from $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t) \text{ is nonempty}\}$ into $\mathcal{L}(X)$

with the induced set ordering $\sqsubseteq$. The existence of increasing optimal solutions, as in Lemma 2.8.1, Theorem 2.8.1, and Theorem 2.8.2, is a fundamental property of complementarity and supermodularity.

The above definition of a collection of sets S_t being "increasing" raises an issue that deserves some comment and clarification. The general definition of an increasing function in Section 2.2 requires that the domain and the range of the function be partially ordered sets. (One might instead state a corresponding broader definition where there is some binary relation on the domain and on the range, but where neither binary relation is necessarily a partial ordering.) When the range of S_t is taken with the induced set ordering $\sqsubseteq$, the antisymmetric and transitive properties for $\sqsubseteq$ require the assumption that each S_t is nonempty and the reflexive property for $\sqsubseteq$ requires the assumption that each S_t is a sublattice. Stating hypotheses or results in terms of a parameterized collection of sets (a correspondence) S_t being increasing in t with respect to $\sqsubseteq$ therefore requires the condition that each S_t be a nonempty sublattice. Accordingly, it is implicitly assumed throughout this monograph that any two sets ordered by the induced set ordering $\sqsubseteq$ are nonempty sublattices and that any collection of sets (correspondence) S_t increasing with a parameter t has each S_t being a nonempty sublattice. One should subsequently take note of where that condition is essential and where it only serves to satisfy a definition. (For example, the assumption that each S_t is a sublattice is not essential for the result of Lemma 2.8.1. However, Example 2.8.6 shows that the result of Theorem 2.8.12 would not hold if this definition did not require that each S_t be a sublattice.)

Theorem 2.4.2, from Topkis [1978], shows that intersecting corresponding members of different increasing collections of parameterized sets (increasing correspondences) yields an increasing collection of parameterized sets (increasing correspondence).

Theorem 2.4.2. *If X is a lattice, T is a partially ordered set, the correspondence $S_{\alpha,t}$ is an increasing function of t from T into $\mathcal{L}(X)$ for each α in a set A, and $\cap_{\alpha\in A}S_{\alpha,t}$ is nonempty for each t in T, then the correspondence $\cap_{\alpha\in A}S_{\alpha,t}$ is an increasing function of t from T into $\mathcal{L}(X)$.*

Proof. Pick any t' and t'' in T with $t' \preceq t''$. Pick any x' in $\cap_{\alpha\in A}S_{\alpha,t'}$ and x'' in $\cap_{\alpha\in A}S_{\alpha,t''}$. Then x' is in $S_{\alpha,t'}$ and x'' is in $S_{\alpha,t''}$ for each α in A, and so $x' \wedge x''$ is in $S_{\alpha,t'}$ and $x' \vee x''$ is in $S_{\alpha,t''}$ for each α in A because $S_{\alpha,t'} \sqsubseteq S_{\alpha,t''}$ by hypothesis. Thus $x' \wedge x''$ is in $\cap_{\alpha\in A}S_{\alpha,t'}$ and $x' \vee x''$ is in $\cap_{\alpha\in A}S_{\alpha,t''}$, so $\cap_{\alpha\in A}S_{\alpha,t'} \sqsubseteq \cap_{\alpha\in A}S_{\alpha,t''}$. $\square$

If X is a lattice, then the closed intervals $[x, \infty)$ and $(-\infty, x]$, are increasing in x on X. The increasing properties of Example 2.4.1 follow from this, together with Theorem 2.4.2 and the earlier observation that $\sqsubseteq$ is reflexive on $\mathcal{L}(X)$.

Example 2.4.1. Suppose that X is a lattice and X' is a sublattice of X.
(a) $X' \cap [x, \infty)$ is increasing in x on $\{x : x \in X, X' \cap [x, \infty)$ is nonempty$\}$.
(b) $X' \cap (-\infty, x]$ is increasing in x on $\{x : x \in X, X' \cap (-\infty, x]$ is nonempty$\}$.
(c) $X' \cap [x', x'']$ is increasing in (x', x'') on $\{(x', x'') : x' \in X, x'' \in X$, and $X' \cap [x', x'']$ is nonempty$\}$.

Example 2.4.2 notes that the direct product of parameterized sets that are increasing correspondences is also an increasing correspondence.

Example 2.4.2. If S_α is a lattice for each α in a set A, T is a partially ordered set, and the correspondence $S_{\alpha t}$ is increasing in t from T into $\mathcal{L}(S_\alpha)$ for each α in A, then the correspondence $\times_{\alpha \in A} S_{\alpha t}$ is increasing in t from T into $\mathcal{L}(\times_{\alpha \in A} S_\alpha)$.

If X and T are sets, S_t is a subset of X for each t in T, and x_t is in S_t for each t in T, then the function x_t from T into X is a **selection** from S_t. If X and T are partially ordered sets, S_t is a subset of X for each t in T, and x_t is a selection from S_t that is an increasing (decreasing) function of t from T into X, then x_t is an **increasing (decreasing) selection**. A useful property of the induced set ordering relation $\sqsubseteq$ is that an increasing correspondence S_t often has an increasing selection with a convenient specification as the greatest (or least) element of each S_t, as shown in Lemma 2.4.2 (from Topkis [1978]) and in Theorem 2.4.3. Theorem 2.4.4 gives another way to specify an increasing selection from an increasing correspondence based on any selection from the sets in the range, where the sets in the range need not have a greatest element or least element but where either the range or the domain is finite. For a collection of optimization problems as in (2.1.1), maximize $f(x, t)$ subject to x in S_t, with each problem determined by a parameter t, an increasing (decreasing) selection from $\text{argmax}_{x \in S_t} f(x, t)$ is an **increasing (decreasing) optimal selection**. The existence of an increasing optimal selection, as in Theorem 2.8.3, is a fundamental property of complementarity and supermodularity.

Lemma 2.4.2. *If X is a lattice, X' and X'' are nonempty subsets of X, $X' \sqsubseteq X''$, and* $\sup_X(X')$ *and* $\sup_X(X'')$ ($\inf_X(X')$ *and* $\inf_X(X'')$) *exist, then* $\sup_X(X') \preceq \sup_X(X'')$ ($\inf_X(X') \preceq \inf_X(X'')$).

Proof. Pick any x' in X' and x'' in X''. Because $X' \sqsubseteq X''$, $x' \wedge x''$ is in X' and $x' \vee x''$ is in X''. If $\sup_X(X')$ and $\sup_X(X'')$ exist, then $x' \preceq x' \vee x'' \preceq \sup_X(X'')$ and so $\sup_X(X') \preceq \sup_X(X'')$. If $\inf_X(X')$ and $\inf_X(X'')$ exist, then $\inf_X(X') \preceq x' \wedge x'' \preceq x''$ and so $\inf_X(X') \preceq \inf_X(X'')$. □

Theorem 2.4.3. *Suppose that X is a lattice, T is a partially ordered set, S_t is a subset of X for each t in T, and S_t is increasing in t on T. If S_t has a greatest* (*least*) *element for each* t *in* T, *then the greatest (least) element is an increasing selection from* S_t. *Hence, if either* S_t *is finite for each* t *in* T *or* X *is a subset of* R^n *and* S_t *is a compact subset of* R^n *for each* t *in* T, *then* S_t *has a greatest element and a least element for each* t *in* T *and the greatest* (*least*) *element is an increasing selection from* S_t.

Proof. The first statement of this result follows from Lemma 2.4.2. The remainder then follows from Lemma 2.2.1 and Corollary 2.3.2, which imply that each S_t has a greatest element and a least element. □

Theorem 2.4.4. *Suppose that* X *is a lattice,* T *is a partially ordered set, either* X *or* T *is finite,* S_t *is a subset of* X *for each* t *in* T, *and* S_t *is increasing in* t *on* T. *If* x'_t *is any selection from* S_t, *then* $\sup\{x'_\tau : \tau \in T, \tau \preceq t\}$ $(\inf\{x'_\tau : \tau \in T, t \preceq \tau\})$ *exists and is an increasing selection from* S_t.

Proof. Because either X or T is finite, $\{x'_\tau : \tau \in T, \tau \preceq t\}$ is finite for each t in T. By Lemma 2.2.1, $\sup\{x'_\tau : \tau \in T, \tau \preceq t\}$ exists for each t in T. Because $\sup\{x'_\tau : \tau \in T, \tau \preceq t\}$ is increasing in t, it suffices to show that $\sup\{x'_\tau : \tau \in T, \tau \preceq t\}$ is a selection; that is, that $\sup\{x'_\tau : \tau \in T, \tau \preceq t\}$ is in S_t for each t in T. Pick any t' in T. For some positive integer k' that depends on t', the elements of $\{x'_\tau : \tau \in T, \tau \preceq t'\}$ can be enumerated $x(1), \ldots, x(k')$ because either X or T is finite. Without loss of generality, suppose that $x(1) = x'_{t'}$ so $x(1)$ is in $S_{t'}$. To complete the proof, it remains to show by induction that $\sup\{x(i) : i = 1, \ldots, k\}$ is in $S_{t'}$ for $k = 1, \ldots, k'$. Because $x(1) = x'_{t'}$ is in $S_{t'}$, $\sup\{x(i) : i = 1, \ldots, k\}$ is in $S_{t'}$ for $k = 1$. Now pick any integer k with $1 \le k < k'$, and suppose that $\sup\{x(i) : i = 1, \ldots, k\}$ is in $S_{t'}$. Pick any τ' in T with $\tau' \preceq t'$ and $x'_{\tau'} = x(k+1)$. Because $\tau' \preceq t'$ and S_t is increasing in t, $S_{\tau'} \sqsubseteq S_{t'}$. Then $x(k+1) = x'_{\tau'}$ in $S_{\tau'}$, $\sup\{x(i) : i = 1, \ldots, k\}$ in $S_{t'}$, and $S_{\tau'} \sqsubseteq S_{t'}$ imply that $\sup\{x(i) : i = 1, \ldots, k+1\} = x(k+1) \vee \sup\{x(i) : i = 1, \ldots, k\}$ is in $S_{t'}$. Hence, by induction, $\sup\{x'_\tau : \tau \in T, \tau \preceq t'\} = \sup\{x(i) : i = 1, \ldots, k'\}$ is in $S_{t'}$. □

Theorem 2.4.5 characterizes the induced set ordering $\sqsubseteq$ and the corresponding property that a correspondence S_t is increasing in t with respect to $\sqsubseteq$ in terms of sections of a sublattice of a product set. For a lattice X and a chain T, a correspondence S_t from T into $\mathcal{P}(X) \setminus \{\emptyset\}$ is increasing in t if and only if there is a sublattice S of $X \times T$ for which each S_t with t in T is the section of S at t. Sufficiency holds more generally when T is any lattice and not just a chain, but Example 2.4.3 shows that necessity does not generally hold for an arbitrary lattice T. Based on Theorem 2.4.5, examples of sublattices yield corresponding examples of increasing correspondences (that is, increasing collections of parameterized sets) and examples of increasing correspondences yield corresponding examples of sublattices. (See Example 2.2.7 and Example 2.4.1.) This connection between sublattices and increasing correspondences elucidates the key role of lattices in monotone comparative statics.

Theorem 2.4.5. *Suppose that X and T are lattices.*
(a) *If S is a sublattice of $X \times T$, then the section S_t of S at t in T is increasing in t on the projection $\Pi_T S$ of S on T.*
(b) *If T is a chain, S_t is a subset of X for each t in T, S_t is increasing in t on $\{t : S_t$ is nonempty$\}$, and $S = \{(x, t) : t \in T, x \in S_t\}$, then S is a sublattice of $X \times T$.*

Proof. Suppose that S is a sublattice of $X \times T$. Pick any t' and t'' in $\Pi_T S$ with $t' \preceq t''$. Now pick x' in $S_{t'}$ and x'' in $S_{t''}$. Because S is a sublattice of $X \times T$, $(x' \vee x'', t'') = (x', t') \vee (x'', t'')$ is in S and $(x' \wedge x'', t') = (x', t') \wedge (x'', t'')$ is in S. Therefore, $x' \vee x''$ is in $S_{t''}$ and $x' \wedge x''$ is in $S_{t'}$, and so $S_{t'} \sqsubseteq S_{t''}$. Thus, S_t is increasing in t on $\Pi_T S$.

Now suppose that T is a chain, S_t is a subset of X for each t in T, S_t is increasing in t on $\{t : S_t$ is nonempty$\}$, and $S = \{(x, t) : t \in T, x \in S_t\}$. Pick any (x', t') and (x'', t'') in S. Without loss of generality since T is a chain, suppose that $t' \preceq t''$. Because x' is in $S_{t'}$, x'' is in $S_{t''}$, and $S_{t'} \sqsubseteq S_{t''}$, $x' \vee x''$ is in $S_{t''}$ and $x' \wedge x''$ in $S_{t'}$. Therefore, $(x', t') \vee (x'', t'') = (x' \vee x'', t'')$ is in S and $(x', t') \wedge (x'', t'') = (x' \wedge x'', t')$ is in S, and so S is a sublattice of $X \times T$. □

Example 2.4.3. Let $X = \{0, 1, 2\}$, $T = \{0, 1\} \times \{0, 1\}$, and $S = \{(0, 0, 0), (1, 0, 1), (1, 1, 0), (2, 1, 1)\}$, so S is a subset of $X \times T$. Then X and T are lattices, S_t is a subset of X for each t in T, and S_t is increasing in t on T, but S is not a sublattice of $X \times T$ because $(1, 0, 1) \vee (1, 1, 0) = (1, 1, 1)$ and $(1, 0, 1) \wedge (1, 1, 0) = (1, 0, 0)$ are not in S.

Lemma 2.4.3 also relates an increasing correspondence to a sublattice.

Lemma 2.4.3. *If X is a lattice, T is a chain, S_t is a subset of X for each t in T, and S_t is increasing in t on T, then $\cup_{t \in T} S_t$ is a sublattice of X.*

Proof. Let $S = \{(x, t) : t \in T, x \in S_t\}$. By part (b) of Theorem 2.4.5, S is a sublattice of $X \times T$. The result then follows from part (b) of Lemma 2.2.3 because $\cup_{t \in T} S_t$ is the projection of S on X. □

Theorem 2.4.6 characterizes the structure of an increasing correspondence for which each sublattice in the range has a greatest element and a least element. The form for this characterization is that of the sufficiency example in part (c) of Example 2.4.1. This result is a variation on a characterization of Milgrom and Shannon [1994] for the structure of two complete sublattices ordered by $\sqsubseteq$.

Theorem 2.4.6. *If X is a lattice, T is a chain, S_t is a subset of X for each t in T, S_t is increasing in t on T, and S_t has a greatest element x''_t and a least element x'_t for each t in T, then there is a sublattice X' of X such that $S_t = X' \cap [x'_t, x''_t]$ for each t in T. Furthermore, $X' = \cup_{t \in T} S_t$.*

Proof. Let $X' = \cup_{t\in T} S_t$. By Lemma 2.4.3, X' is a sublattice of X. Pick any t in T. Clearly S_t is a subset of $X' \cap [x'_t, x''_t]$. Now pick any x in $X' \cap [x'_t, x''_t]$. Because x is in X', x is in $S_{t'}$ for some t' in T. Because T is a chain, either $t \preceq t'$ or $t' \preceq t$. If $t \preceq t'$, then $x \preceq x''_t$ and $S_t \sqsubseteq S_{t'}$ imply that $x = x''_t \wedge x$ is in S_t. If $t' \preceq t$, then $x'_t \preceq x$ and $S_{t'} \sqsubseteq S_t$ imply that $x = x \vee x'_t$ is in S_t. Thus, $X' \cap [x'_t, x''_t]$ is a subset of S_t. □

If X is a partially ordered set, then the **weak induced set ordering $\sqsubseteq_w$** on $\mathcal{P}(X) \setminus \{\emptyset\}$ is defined such that $X' \sqsubseteq_w X''$ in $\mathcal{P}(X) \setminus \{\emptyset\}$ if x' in X' implies that there exists x'' in X'' with $x' \preceq x''$ and if x'' in X'' implies that there exists x' in X' with $x' \preceq x''$. That is, $X' \sqsubseteq_w X''$ in $\mathcal{P}(X) \setminus \{\emptyset\}$ if $X'' \cap [x', \infty)$ is nonempty for each x' in X' and if $X' \cap (-\infty, x'']$ is nonempty for each x'' in X''. The weak induced set ordering is a natural notion of order among partially ordered sets. If X is a lattice, then the induced set ordering $\sqsubseteq$ is certainly stronger than the weak induced set ordering $\sqsubseteq_w$. However, Theorem 2.4.7 characterizes the induced set ordering in terms of the weak induced set ordering, further motivating the use of the induced set ordering as an expression of order among sets. The weak induced set ordering $\sqsubseteq_w$ is reflexive and transitive on $\mathcal{P}(X) \setminus \{\emptyset\}$, but it is not antisymmetric as can be seen from $\{0, 1, 3\} \sqsubseteq_w \{0, 2, 3\}$ and $\{0, 2, 3\} \sqsubseteq_w \{0, 1, 3\}$.

Theorem 2.4.7. *If X is a lattice, X' and X'' are nonempty sublattices of X, and $X' \cap [x', x''] \sqsubseteq_w X'' \cap [x', x'']$ for all x' and x'' in X with $X' \cap [x', x'']$ and $X'' \cap [x', x'']$ nonempty, then $X' \sqsubseteq X''$.*

Proof. Pick any z' in X' and z'' in X''. Let $x' = z' \wedge z''$ and $x'' = z' \vee z''$, so $X' \cap [x', x'']$ and $X'' \cap [x', x'']$ are nonempty.

Because z' is in $X' \cap [x', x'']$ and $X' \cap [x', x''] \sqsubseteq_w X'' \cap [x', x'']$, there exists x in $X'' \cap [x', x'']$ with $z' \preceq x$. Because X'' is a sublattice of X and $z' \preceq x \preceq x'' = z' \vee z''$, $z' \vee z'' = x \vee z''$ is in X''.

Because z'' is in $X'' \cap [x', x'']$ and $X' \cap [x', x''] \sqsubseteq_w X'' \cap [x', x'']$, there exists x in $X' \cap [x', x'']$ with $x \preceq z''$. Because X' is a sublattice of X and $z' \wedge z'' = x' \preceq x \preceq z''$, $z' \wedge z'' = z' \wedge x$ is in X'. □

2.5 Fixed Points

If $f(x)$ is a function from a set X into X and if x' in X satisfies $f(x') = x'$, then x' is a **fixed point** of $f(x)$. If $f(x)$ is a correspondence from a set X into the power set $\mathcal{P}(X)$ and if x' in X is in $f(x')$, then x' is a **fixed point** of $f(x)$. The second definition is equivalent to the first when each $f(x)$ is a singleton. Tarski [1955] shows that the collection of fixed points of an increasing function from a nonempty complete lattice into itself is a nonempty complete lattice, and he gives the form of the greatest fixed point and the least fixed point. Conversely, Davis [1955] establishes that if every increasing function from a lattice into itself has a fixed point, then the lattice is complete. Zhou [1994] extends Tarski's [1955] result to correspondences. See also Fujimoto [1984]. Theorem 2.5.1 gives Zhou's [1994] fixed point result, with Tarski's [1955] result following in Corollary 2.5.1.

Results on fixed points are relevant for the study of noncooperative games because the set of all equilibrium points in a noncooperative game is identical to the set of fixed points of a related correspondence. (See Lemma 4.2.1.) Theorem 2.5.1 is useful in establishing the existence of pure equilibrium points in certain noncooperative games. (See Theorem 4.2.1.)

Theorem 2.5.1. *Suppose that X is a nonempty complete lattice, $Y(x)$ is an increasing correspondence from X into $\mathcal{L}(X)$ with $\mathcal{L}(X)$ having the induced set ordering $\sqsubseteq$, and $Y(x)$ is subcomplete for each x in X.*
(a) *The set of fixed points of $Y(x)$ in X is nonempty,* $\sup_X(\{x : x \in X, Y(x) \cap [x, \infty)$ *is nonempty*$\})$ *is the greatest fixed point, and* $\inf_X(\{x : x \in X, Y(x) \cap (-\infty, x]$ *is nonempty*$\})$ *is the least fixed point.*
(b) *The set of fixed points of $Y(x)$ in X is a nonempty complete lattice.*

Proof. Let $X' = \{x : x \in X, Y(x) \cap [x, \infty) \text{ is nonempty}\}$. Because X is a nonempty complete lattice, $\inf_X(X)$ exists. Because $Y(\inf_X(X))$ is in $\mathcal{L}(X)$, $Y(\inf_X(X))$ is a nonempty subset of X. Therefore, $\inf_X(X)$ is in X' and so X' is nonempty. Let $x' = \sup_X(X')$, which exists because X is a complete lattice and X' is a nonempty subset of X. Let y' be the greatest element of $Y(x')$, which exists because $Y(x')$ is a nonempty subcomplete sublattice. If x is in X', then there exists x'' in $Y(x)$ with $x \preceq x'' \preceq \sup_X(Y(x)) = y'$ and so $x' = \sup_X(X') \preceq y'$. Because $Y(x)$ is increasing, $Y(x') \sqsubseteq Y(y')$. Let z' be the greatest element in $Y(y')$, so $y' \preceq z'$ by Lemma 2.4.2. Therefore, z' is in $Y(y') \cap [y', \infty)$ and so y' is in X'. Thus, $y' \preceq \sup_X(X') = x' \preceq y'$, and so $y' = x' = \sup_X(X')$ is the greatest element of X' and is also the greatest element of $Y(x')$. Because x' is in $Y(x')$, x' is a fixed point. Because each fixed point is in X' and the fixed point x' is the greatest element of X', $x' = \sup_X(\{x : x \in X, Y(x) \cap [x, \infty) \text{ is nonempty}\})$ is the greatest fixed point. By now considering the dual of X, $\inf_X(\{x : x \in X, Y(x) \cap (-\infty, x] \text{ is nonempty}\})$ is the least fixed point. This completes the proof of part (a).

Pick any nonempty set X'' of fixed points. Recall that the intersection of a complete lattice (subcomplete sublattice) and a closed interval is complete (subcomplete). Because X is a complete lattice, $\sup_X(X'')$ exists and $X \cap [\sup_X(X''), \infty)$ is a nonempty complete lattice. If x' is in X'', then x' is in $Y(x')$ because X'' is a set of fixed points, and so $x' \preceq \sup(Y(x')) \preceq \sup(Y(\sup_X(X'')))$ by Lemma 2.4.2 because $Y(x)$ increasing implies $Y(x') \sqsubseteq Y(\sup_X(X''))$. Thus, $\sup_X(X'') \preceq \sup(Y(\sup_X(X'')))$, and so if x' is in $X \cap [\sup_X(X''), \infty)$ then $\sup_X(X'') \preceq \sup(Y(\sup_X(X''))) \preceq \sup(Y(x'))$ by Lemma 2.4.2 because $Y(x)$ increasing implies $Y(\sup_X(X'')) \sqsubseteq Y(x')$. For x in X, let $Z(x) = Y(x) \cap [\sup_X(X''), \infty)$. For x in X, $Z(x)$ is a subcomplete sublattice because $Y(x)$ is a subcomplete sublattice. For x in $X \cap [\sup_X(X''), \infty)$, $\sup(Y(x))$ is in $Z(x)$ and so $Z(x)$ is nonempty. By Theorem 2.4.2, $Z(x)$ is an increasing correspondence from $X \cap [\sup_X(X''), \infty)$ into $\mathcal{L}(X \cap [\sup_X(X''), \infty))$. By part (a), the least fixed point of $Z(x)$ on $X \cap [\sup_X(X''), \infty)$ exists and is $y' = \inf_{X \cap [\sup_X(X''), \infty)}(\{x : x \in X \cap [\sup_X(X''), \infty), Z(x) \cap (-\infty, x] \text{ is nonempty}\})$. Let X''' be the set of all fixed points of $Y(x)$ on X. Because y' is a fixed point of $Z(x)$ on $X \cap [\sup_X(X''), \infty)$, y' is in X'''. Let y'' be any element of X''' that is an upper bound for X''. Then y'' is in $\{x : x \in X \cap [\sup_X(X''), \infty), Z(x) \cap (-\infty, x] \text{ is nonempty}\}$, so $y' \preceq y''$ and $y' = \sup_{X'''}(X'')$. By now considering the dual of X, $\inf_{X'''}(X'')$ exists. This completes the proof of part (b) that X''' is a complete lattice. □

When each $Y(x)$ is a singleton, the hypotheses on $Y(x)$ in Theorem 2.5.1 reduce to $Y(x)$ being an increasing function from X into X. With this observation, Theorem 2.5.1 yields Corollary 2.5.1.

Corollary 2.5.1. *Suppose that $f(x)$ is an increasing function from a nonempty complete lattice X into X.*
(a) *The set of fixed points of $f(x)$ in X is nonempty, $\sup_X(\{x : x \in X, x \preceq f(x)\})$ is the greatest fixed point, and $\inf_X(\{x : x \in X, f(x) \preceq x\})$ is the least fixed point.*
(b) *The set of fixed points of $f(x)$ in X is a nonempty complete lattice.*

While the set of fixed points is a complete lattice under the hypotheses of Theorem 2.5.1 and Corollary 2.5.1, Example 2.5.1 shows that these results cannot be extended to show that the set of fixed points of an increasing function from a nonempty complete lattice X into itself is a sublattice of X.

Example 2.5.1. Let X be the closed interval $[(0,0), (3,3)]$ in R^2. Define $f(x)$ on X so that $f(1,2) = (1,2)$, $f(2,1) = (2,1)$, $f(x) = (0,0)$ if $x_1 + x_2 < 3$, and $f(x) = (3,3)$ if $x_1 + x_2 \geq 3$, $x \neq (1,2)$, and $x \neq (2,1)$. Then $f(x)$ is an increasing function from the nonempty complete lattice X into X, but the set of fixed points $\{(0,0),(2,1),(1,2),(3,3)\}$ is not a sublattice of X. (See Example 2.2.4.)

Milgrom and Roberts [1994] give conditions for a parameterized function from a nonempty complete lattice into itself to have greatest and least fixed points that are increasing with the parameter. Theorem 2.5.2 extends this to correspondences, with the result of Milgrom and Roberts [1994] following in Corollary 2.5.2. Theorem 2.5.2 is useful in showing that the greatest and least equilibrium points in certain parameterized noncooperative games are increasing with the parameter. (See Theorem 4.2.2.)

Theorem 2.5.2. *Suppose that X is a nonempty complete lattice, T is a partially ordered set, $Y(x,t)$ is a nonempty subcomplete sublattice of X for each (x,t) in $X \times T$, and the correspondence $Y(x,t)$ (with its range having the induced set ordering $\sqsubseteq$ on $\mathcal{L}(X)$) is an increasing function on $X \times T$.*
(a) *For each t in T, there exists a greatest (least) fixed point of $Y(x,t)$.*
(b) *The greatest (least) fixed point of $Y(x,t)$ is increasing in t on T.*
(c) *If, in addition, $\sup Y(x',t') \prec \inf Y(x',t'')$ for all x' in X and $t' \prec t''$ in T, then the greatest (least) fixed point of $Y(x,t)$ is strictly increasing in t on T.*

Proof. Part (a) is immediate from part (a) of Theorem 2.5.1.

Pick any t' and t'' in T with $t' \prec t''$. Let $x_{t'}$ and $x_{t''}$ be the greatest fixed points for $Y(x,t')$ and $Y(x,t'')$, respectively. Because $x_{t'}$ is in $Y(x_{t'},t')$ and $Y(x_{t'},t') \sqsubseteq Y(x_{t'},t'')$, $Y(x_{t'},t'') \cap [x_{t'},\infty)$ is nonempty. By part (a) of Theorem 2.5.1, $x_{t'} \preceq x_{t''}$.

Now suppose that, in addition, $\sup Y(x',t') \prec \inf Y(x',t'')$ for each x' in X. Then

$$x_{t'} \preceq \sup Y(x_{t'},t') \prec \inf Y(x_{t'},t'') \preceq \inf Y(x_{t''},t'') \preceq x_{t''},$$

where the first and last inequalities follow because $x_{t'}$ and $x_{t''}$ are fixed points of $Y(x,t')$ and $Y(x,t'')$, respectively, and the third inequality follows from Lemma 2.4.2 because $x_{t'} \preceq x_{t''}$.

The proofs of part (b) and part (c) for the least fixed points are similar and so are omitted. □

For the case where each $Y(x,t)$ is a singleton, the statement of Theorem 2.5.2 reduces to that of Corollary 2.5.2.

Corollary 2.5.2. *Suppose that X is a nonempty complete lattice, T is a partially ordered set, and $f(x,t)$ is an increasing function from $X \times T$ into X.*
(a) *For each t in T, there exists a greatest (least) fixed point of $f(x,t)$.*
(b) *The greatest (least) fixed point of $f(x,t)$ is increasing in t on T.*

(c) *If, in addition, $f(x, t)$ is strictly increasing in t on T for each x in X, then the greatest (least) fixed point of $f(x, t)$ is strictly increasing in t on T.*

2.6 Supermodular Functions on a Lattice

This section introduces supermodular functions and some of their basic properties. Subsection 2.6.1 characterizes supermodularity in terms of increasing differences and complementarity. Subsection 2.6.2 considers a variety of transformations that maintain or generate supermodularity. Subsection 2.6.3 looks at ordinal generalizations of the cardinal notions of supermodularity and increasing differences. Subsection 2.6.4 discusses log-supermodularity, which is closely related to the monotonicity of price elasticity.

2.6.1 Characterization and Complementarity

Suppose that X and T are partially ordered sets and $f(x, t)$ is a real-valued function on a subset S of $X \times T$. For t in T, let S_t denote the section of S at t. If $f(x, t'') - f(x, t')$ is increasing, decreasing, strictly increasing, or strictly decreasing in x on $S_{t''} \cap S_{t'}$ for all $t' \prec t''$ in T, then $f(x, t)$ has, respectively, **increasing differences**, **decreasing differences**, **strictly increasing differences**, or **strictly decreasing differences** in (x, t) on S. The conditions of these definitions do not distinguish between the first and second variables because $f(x', t'') - f(x', t') \le f(x'', t'') - f(x'', t')$ if and only if $f(x'', t') - f(x', t') \le f(x'', t'') - f(x', t'')$, and similarly for a strict inequality.

Suppose that X_α is a partially ordered set for each α in a set A, X is a subset of $\times_{\alpha \in A} X_\alpha$, an element x in X is expressed as $x = (x_\alpha : \alpha \in A)$ where x_α is in X_α for each α in A, and $f(x)$ is a real-valued function on X. If, for all distinct α' and α'' in A and for all x'_α in X_α for all α in $A \setminus \{\alpha', \alpha''\}$, $f(x)$ has increasing differences, decreasing differences, strictly increasing differences, or strictly decreasing differences in $(x_{\alpha'}, x_{\alpha''})$ on the section of X at $\{x'_\alpha : \alpha \in A \setminus \{\alpha', \alpha''\}\}$, then $f(x)$ has, respectively, **increasing differences**, **decreasing differences**, **strictly increasing differences**, or **strictly decreasing differences** on X. If $f(x)$ is differentiable on R^n, then $f(x)$ has increasing differences on R^n if and only if $\partial f(x)/\partial x_{i'}$ is increasing in $x_{i''}$ for all distinct i' and i'' and all x. If $f(x)$ is twice differentiable on R^n, then $f(x)$ has increasing differences on R^n if and only if $\partial^2 f(x)/\partial x_{i'} \partial x_{i''} \ge 0$ for all distinct i' and i'' and all x.

Increasing differences is a well-known condition for a utility function to be that of a system of complementary products (Edgeworth [1925]; Samuelson [1947, 1974]). Suppose that $f(x)$ is the real-valued utility function (or minus the cost function) for a system of n products whose levels are $x = (x_1, \ldots, x_n)$. Then, recalling that u^i denotes the i^{th} unit vector in R^n, $f(x + \epsilon u^i) - f(x)$

is the additional utility for an additional $\epsilon > 0$ units of product i. The utility function $f(x)$ has increasing differences if and only if the net additional utility for any additional amount of each product i' is always increasing in the level of every other product i''; that is, the desirability of more product i' always increases with the amount available of every other product i''. Hence, a collection of products (or activities or other decision variables or parameters) are **complements** and each pair is said to be **complementary** if the products have a real-valued utility function (or minus a cost function) with increasing differences. (Bulow, Geanakoplos, and Klemperer [1985] use the term **strategic complements** to describe activities i' and i'' with $\partial^2 f(x)/\partial x_{i'} \partial x_{i''} \geq 0$ for all x where $f(x)$ is a twice differentiable utility function on R^n.) If S is a set of activities (or other decision variables or parameters) that are potentially available, $g(X)$ is the real-valued utility function (or minus the cost function) on the power set $\mathcal{P}(S)$ for having a subset X of activities available, $f(x)$ is defined on $\times_{s \in S}\{0, 1\}$ such that $f(1(X)) = g(X)$ for each subset X of S (where $1(X)$ is the indicator vector of X), and $f(x)$ has increasing differences on $\times_{s \in S}\{0, 1\}$, then the elements of the set S are **complements** and each pair is said to be **complementary**. That is, a set of activities are complements if the additional utility resulting from the availability of any additional activity is increasing with the set of other activities available (where sets of activities are ordered by set inclusion). The present definition is but one of a number of different notions that may be used to formalize the concept of complementarity (Samuelson [1947, 1974]). Equivalences and distinctions between some of these concepts are summarized in Section 2.9.

Interpretations and definitions for **substitutes** are similar to those given above for complements, but in terms of utility functions (or minus the cost functions) that have decreasing differences rather than increasing differences.

Suppose that $f(x)$ is a real-valued function on a lattice X. If

$$f(x') + f(x'') \leq f(x' \vee x'') + f(x' \wedge x'')$$

for all x' and x'' in X, then $f(x)$ is **supermodular** on X. If

$$f(x') + f(x'') < f(x' \vee x'') + f(x' \wedge x'')$$

for all unordered x' and x'' in X, then $f(x)$ is **strictly supermodular** on X. If $-f(x)$ is (strictly) supermodular, then $f(x)$ is (**strictly**) **submodular**. A function that is both supermodular and submodular is a **valuation**.

Theorem 2.6.1 and Corollary 2.6.1 show that a function has increasing differences on the direct product of a finite collection of chains if and only if the function is supermodular on that direct product, thereby characterizing supermodularity on the direct product of a finite collection of chains (and, in particular, on R^n) in terms of the nonnegativity of all pairs of cross-differences. The number 2 thus has a fundamental role for supermodular functions, since supermodularity on any finite product of chains is equivalent to supermodularity on the product of each pair of the chains. (The number 2 likewise plays a fundamental role in characterizing sublattice structure, as in Lemma 2.2.4, Theorem 2.2.1, and Theorem 2.2.2.) Thus supermodularity, like concavity (Rockafellar [1970]), is a second-order property in the sense that for twice-differentiable functions on R^n each class of functions can be characterized by certain conditions on the matrix of second partial derivatives. It is usually more natural to approach and understand applications in terms of increasing differences and complementarity, which are more economically meaningful concepts. However, the attendant mathematical analyses tend to be easier to handle in terms of supermodularity. Theoretical connections between supermodularity and complementarity are further developed in the remainder of this chapter, with applications of these connections presented in Chapter 3, Chapter 4, and Chapter 5.

Theorem 2.6.1, from Topkis [1978], shows that supermodularity implies increasing differences for a function on a sublattice of the direct product of lattices.

Theorem 2.6.1. *If X_α is a lattice for each α in a set A, X is a sublattice of $\times_{\alpha \in A} X_\alpha$, and $f(x)$ is (strictly) supermodular on X, then $f(x)$ has (strictly) increasing differences on X.*

Proof. Suppose that $f(x)$ is supermodular on X. (The proof for the case with $f(x)$ strictly supermodular is similar.) Pick any distinct α' and α'' in A and any x'_α in X_α for each α in $A \setminus \{\alpha', \alpha''\}$. For any x in X with $x_\alpha = x'_\alpha$ for each α in $A \setminus \{\alpha', \alpha''\}$, define $g(x_{\alpha'}, x_{\alpha''}) = f(x)$. It suffices to show that $g(x_{\alpha'}, x_{\alpha''})$ has increasing differences in $(x_{\alpha'}, x_{\alpha''})$ on the section of X at $\{x'_\alpha : \alpha \in A \setminus \{\alpha', \alpha''\}\}$. Pick any $x'_{\alpha'}$ and $x''_{\alpha'}$ in $X_{\alpha'}$ and $x'_{\alpha''}$ and $x''_{\alpha''}$ in $X_{\alpha''}$ such that $x'_{\alpha'} \preceq x''_{\alpha'}$, $x'_{\alpha''} \preceq x''_{\alpha''}$, and $\{(x'_{\alpha'}, x'_{\alpha''}), (x'_{\alpha'}, x''_{\alpha''}), (x''_{\alpha'}, x'_{\alpha''}), (x''_{\alpha'}, x''_{\alpha''})\}$ is a subset of the section of X at $\{x'_\alpha : \alpha \in A \setminus \{\alpha', \alpha''\}\}$. By the supermodularity of $f(x)$,

$$\begin{aligned} g(x'_{\alpha'}, x''_{\alpha''}) - g(x'_{\alpha'}, x'_{\alpha''}) &= g(x'_{\alpha'}, x''_{\alpha''}) - g(x'_{\alpha'} \wedge x''_{\alpha'}, x'_{\alpha''} \wedge x''_{\alpha''}) \\ &\le g(x'_{\alpha'} \vee x''_{\alpha'}, x'_{\alpha''} \vee x''_{\alpha''}) - g(x''_{\alpha'}, x'_{\alpha''}) \\ &= g(x''_{\alpha'}, x''_{\alpha''}) - g(x''_{\alpha'}, x'_{\alpha''}). \quad \square \end{aligned}$$

Theorem 2.6.2 gives a converse of Theorem 2.6.1, showing that on the direct product of finitely many lattices increasing differences together with supermodularity in each component implies supermodularity.

Theorem 2.6.2. *If X_i is a lattice for $i = 1, \dots, n$, $f(x)$ has (strictly) increasing differences on $\times_{i=1}^n X_i$, and $f(x)$ is (strictly) supermodular in $x_{i'}$ on $X_{i'}$ for all fixed x_i in X_i for all $i \neq i'$ and for each $i' = 1, \dots, n$, then $f(x)$ is (strictly) supermodular on $\times_{i=1}^n X_i$.*

Proof. Suppose that $f(x)$ has increasing differences on $\times_{i=1}^n X_i$ and is supermodular in x_i on X_i for each i. (The proof for the case with $f(x)$ having strictly increasing differences and being strictly supermodular in each component is similar.) Pick any x' and x'' in $\times_{i=1}^n X_i$. Then, with the first inequality following from increasing differences and the second inequality following from supermodularity,

$$\begin{aligned}
&f(x') - f(x' \wedge x'') \\
&\quad = \textstyle\sum_{i=1}^n (f(x'_1, \dots, x'_i, x'_{i+1} \wedge x''_{i+1}, \dots, x'_n \wedge x''_n) \\
&\qquad\qquad - f(x'_1, \dots, x'_{i-1}, x'_i \wedge x''_i, \dots, x'_n \wedge x''_n)) \\
&\quad \le \textstyle\sum_{i=1}^n (f(x'_1 \vee x''_1, \dots, x'_{i-1} \vee x''_{i-1}, x'_i, x''_{i+1}, \dots, x''_n) \\
&\qquad\qquad - f(x'_1 \vee x''_1, \dots, x'_{i-1} \vee x''_{i-1}, x'_i \wedge x''_i, x''_{i+1}, \dots, x''_n)) \\
&\quad \le \textstyle\sum_{i=1}^n (f(x'_1 \vee x''_1, \dots, x'_{i-1} \vee x''_{i-1}, x'_i \vee x''_i, x''_{i+1}, \dots, x''_n) \\
&\qquad\qquad - f(x'_1 \vee x''_1, \dots, x'_{i-1} \vee x''_{i-1}, x''_i, x''_{i+1}, \dots, x''_n)) \\
&\quad = f(x' \vee x'') - f(x'')
\end{aligned}$$

and so $f(x)$ is supermodular on $\times_{i=1}^n X_i$. □

Corollary 2.6.1, from Topkis [1978], states that increasing differences implies supermodularity on the direct product of a finite collection of chains. This result is a consequence of Theorem 2.6.2 because any real-valued function on a chain is (strictly) supermodular. (See part (a) of Example 2.6.2.) Whitney [1935] proves a related result involving the rank function of a matroid, which is a submodular set function.

Corollary 2.6.1. *If X_i is a chain for $i = 1, \dots, n$ and $f(x)$ has (strictly) increasing differences on $\times_{i=1}^n X_i$, then $f(x)$ is (strictly) supermodular on $\times_{i=1}^n X_i$.*

Example 2.6.1, from Topkis [1978], shows that increasing differences does not imply supermodularity on the direct product of a countable collection of chains, so the result of Corollary 2.6.1 cannot be extended to hold for the direct product of more than finitely many chains.

Example 2.6.1. Let $X_i = \{0, 1\}$ for $i = 1, 2, \dots$, and $x = (x_1, x_2, \dots)$ where x_i is in X_i for $i = 1, 2, \dots$. Define $f(x)$ on $\times_{i=1}^\infty X_i$ so that $f(x) = 1$ if $x_i = 1$ for an infinite set of indices i and $f(x) = 0$ if $x_i = 1$ for only a finite set of indices i. With respect to any finite set of components of x, $f(x)$ is constant. Therefore, $f(x)$ has both increasing differences and decreasing differences on $\times_{i=1}^{\infty} X_i$. However, if x' and x'' are defined such that $x'_i = 1$ for i odd, $x'_i = 0$ for i even, $x''_i = 1$ for i even, and $x''_i = 0$ for i odd, then

$$f(x') + f(x'') = 2 > 1 = f(x' \vee x'') + f(x' \wedge x'')$$

and so $f(x)$ is not supermodular on $\times_{i=1}^{\infty} X_i$.

Example 2.6.2 gives examples of supermodular functions. The supermodularity conclusion for each part except part (b) is based on observing increasing differences and applying Corollary 2.6.1.

Example 2.6.2. The following are examples of supermodular functions.
(a) If X is a chain, then any real-valued function on X is a valuation (that is, both supermodular and submodular). Therefore, any real-valued function on a subset of R^1 is a valuation.
(b) If S is a set, the power set $\mathcal{P}(S)$ is taken with the set inclusion ordering relation $\subseteq$, $\mathcal{X}$ is a sublattice of $\mathcal{P}(S)$, and $f(X)$ is a real-valued function on $\mathcal{X}$ such that $f(X') + f(X'') \leq f(X' \cup X'') + f(X' \cap X'')$ for all X' and X'' in $\mathcal{X}$, then $f(X)$ is supermodular on $\mathcal{X}$. (Recall part (e) of Example 2.2.3.) For a set S, a supermodular function on a sublattice of $\mathcal{P}(S)$ is equivalent to a supermodular function on a sublattice of $\times_{s \in S}\{0, 1\}$ by the transformation of part (d) of Example 2.2.1. Hence, for a finite set S, a supermodular function on a sublattice of $\mathcal{P}(S)$ is equivalent to a supermodular function on a sublattice of $R^{|S|}$.
(c) The function $f(x) = x_1 x_2$ is supermodular on R^2.
(d) For x and t in R^n, $x \cdot t = \sum_{i=1}^{n} x_i t_i$ is supermodular in (x, t) on R^{2n}.
(e) If $\alpha_i \geq 0$ for $i = 1, \ldots, n$, the **Cobb-Douglas** function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is supermodular in x on $\{x : x \in R^n, x \geq 0\}$.
(f) The utility function for **perfect complements**, $f(x) = \min\{\alpha_i x_i : i = 1, \ldots, n\}$ where $\alpha_i \geq 0$ ($\alpha_i \leq 0$) for $i = 1, \ldots, n$, is supermodular on R^n. More generally, if $f_i(z)$ is increasing (decreasing) on R^1 for $i = 1, \ldots, n$ then $f(x) = \min\{f_i(x_i) : i = 1, \ldots, n\}$ is supermodular on R^n. Likewise, if α_i is in R^1 for $i = 1, \ldots, n$, then $f(S) = \min_{i \in S} \alpha_i$ is supermodular in S for subsets S of $\{1, \ldots, n\}$.
(g) The function $f(x, z) = -\sum_{i=1}^{n} |x_i - z_i|$ is supermodular in (x, z) on R^{2n}.
(h) The function $f(x, z) = -|x - z|^2 = -\sum_{i=1}^{n}(x_i - z_i)^2$ is supermodular in (x, z) on R^{2n}. (However, by Theorem 2.6.1, $f(x, z) = -|x - z| = -(\sum_{i=1}^{n}(x_i - z_i)^2)^{1/2}$ is not supermodular in (x, z) on R^{2n} for $n \geq 2$.)

(i) If N is a set, Q is a fixed subset of N, $f_Q(S) = 1$ for any subset S of N with Q being a subset of S, and $f_Q(S) = 0$ for any subset S of N with $Q \setminus S$ nonempty, then $f_Q(S)$ is supermodular in S on the collection $\mathcal{P}(N)$ of all subsets of N.

Because any real-valued function on R^1 is supermodular as in part (a) of Example 2.6.2, supermodular functions on R^n are not generally endowed with continuity and differentiability properties. This contrasts with the strong continuity and differentiability properties of concave functions on R^n (Rockafellar [1970]).

The result of Corollary 2.6.1, characterizing supermodularity in terms of increasing differences, is limited to domains that are the direct product of finitely many chains. Theorem 2.6.3 extends this result to provide a characterization of supermodular functions in terms of increasing differences on an arbitrary sublattice of the direct product of finitely many chains. Note that Corollary 2.6.1 states that a function is supermodular on the direct product of finitely many chains if the function is supermodular on each 4-element sublattice $\{x', x'', x' \vee x'', x' \wedge x''\}$ such that x' and x'' each differ from $x' \wedge x''$ in exactly one component. Recall from Subsection 2.2 that the latter condition is equivalent to x' and x'' each corresponding to increasable covers for $x' \wedge x''$. This observation points to the nature of the generalization in Theorem 2.6.3, which essentially broadens the increasing differences requirement from being with respect to all pairs of distinct components to being with respect to all pairs of distinct increasable covers.

Theorem 2.6.3. *If X_i is a chain for $i = 1, \ldots, n$, X is a sublattice of $\times_{i=1}^n X_i$, and $f(x)$ is (strictly) supermodular on $\{x', x'', x' \vee x'', x' \wedge x''\}$ for all unordered pairs x' and x'' in X such that*

$$X \cap (\times_{i=1}^n \{x'_i, x''_i\}) = \{x', x'', x' \vee x'', x' \wedge x''\},$$

then $f(x)$ is (strictly) supermodular on X.

Proof. Suppose that $f(x)$ is not supermodular on X. (The proof for the case of $f(x)$ being strictly supermodular is similar.) Then there exist unordered x' and x'' in X with

$$0 < f(x') + f(x'') - f(x' \vee x'') - f(x' \wedge x'').$$

Among all such x' and x'', select a pair x' and x'' that maximize the number of indices i with $x'_i = x''_i$. Suppose that

$$X \cap (\times_{i=1}^n \{x'_i, x''_i\}) \neq \{x', x'', x' \vee x'', x' \wedge x''\}.$$

By part (c) of Theorem 2.2.3, there exist z' and z'' in $X \cap (\times_{i=1}^{n}\{x'_i, x''_i\})$ such that

$$X \cap (\times_{i=1}^{n}\{z'_i, z''_i\}) = \{z', z'', z' \vee z'', z' \wedge z''\},$$

z' and z'' correspond to distinct increasable covers for $x' \wedge x''$ in $X \cap (\times_{i=1}^{n}\{x'_i, x''_i\})$, $z' \preceq x'$, $z'' \preceq x''$, $x' \prec x' \vee z''$, and $x'' \prec x'' \vee z'$. Therefore, $x' \wedge x'' \prec z' \preceq x'$, $x' \wedge x'' \prec z'' \preceq x''$, and either $z' \prec x'$ or $z'' \prec x''$. Suppose that $z' \prec x'$. (The proof for the case with $z'' \prec x''$ is similar, and so it is omitted.)

Observing that $x' \wedge x'' = z' \wedge x''$,

$$\begin{aligned} 0 &< f(x') + f(x'') - f(x' \vee x'') - f(x' \wedge x'') \\ &= (f(z') + f(x'') - f(z' \vee x'') - f(z' \wedge x'')) \\ &\quad + (f(x') + f(z' \vee x'') - f(x' \vee x'') - f(z')). \end{aligned}$$

Therefore, either

$$0 < f(z') + f(x'') - f(z' \vee x'') - f(z' \wedge x'')$$

or

$$0 < f(x') + f(z' \vee x'') - f(x' \vee x'') - f(z').$$

For each of these two cases, the pair z' and x'' and the pair x' and $z' \vee x''$ have more of their components equal than the pair x' and x''. But this contradicts the choice of x' and x''. Therefore,

$$X \cap (\times_{i=1}^{n}\{x'_i, x''_i\}) = \{x', x'', x' \vee x'', x' \wedge x''\}. \quad \square$$

Theorem 2.6.4, from Topkis [1978], characterizes those functions that are both supermodular and submodular on the direct product of finitely many chains as the separable functions. Correspondingly, an **affine function** is a function that is the sum of a constant and a linear function, and a real-valued function is both concave and convex on R^n if and only if it is an affine function (Rockafellar [1970]).

Theorem 2.6.4. *If X_i is a chain for $i = 1, \ldots, n$, then a real-valued function $f(x)$ is separable on $\times_{i=1}^{n} X_i$ if and only if $f(x)$ is a valuation on $\times_{i=1}^{n} X_i$.*

Proof. Suppose that $f(x)$ is separable, so $f(x) = \sum_{i=1}^{n} f_i(x_i)$ where $f_i(x_i)$ is a real-valued function on X_i for $i = 1, \dots, n$. Pick any x' and x'' in $\times_{i=1}^{n} X_i$. Because each X_i is a chain, $\{x'_i \vee x''_i, x'_i \wedge x''_i\} = \{x'_i, x''_i\}$ for each i with $1 \le i \le n$. Therefore,

$$\begin{aligned} f(x' \vee x'') + f(x' \wedge x'') &= \textstyle\sum_{i=1}^{n}(f_i(x'_i \vee x''_i) + f_i(x'_i \wedge x''_i)) \\ &= \textstyle\sum_{i=1}^{n}(f_i(x'_i) + f_i(x''_i)) \\ &= f(x') + f(x''). \end{aligned}$$

Hence, $f(x)$ is a valuation on $\times_{i=1}^{n} X_i$.

Now suppose that $f(x)$ is a valuation on $\times_{i=1}^{n} X_i$. By Theorem 2.6.1, $f(x)$ has both increasing differences and decreasing differences on $\times_{i=1}^{n} X_i$. Pick some fixed x' in $\times_{i=1}^{n} X_i$. Now choose any x in $\times_{i=1}^{n} X_i$. Then

$$\begin{aligned} f(x) = f(x') + \textstyle\sum_{i=1}^{n}(f(x_1, \dots, x_i, x'_{i+1}, \dots, x'_n) \\ - f(x_1, \dots, x_{i-1}, x'_i, \dots, x'_n)) \\ = f(x') + \textstyle\sum_{i=1}^{n}(f(x'_1, \dots, x'_{i-1}, x_i, x'_{i+1}, \dots, x'_n) - f(x')), \end{aligned}$$

which is separable. □

2.6.2 Transformations

Lemma 2.6.1, from Topkis [1978], follows directly from the definition of a supermodular function. This result asserts that the set of all supermodular functions on a lattice X is a closed convex cone in the vector space of all real-valued functions on X. Correspondingly, the set of all concave functions on a convex set X is a closed convex cone in the vector space of all real-valued functions on X. A similar result holds for functions with increasing differences. Versions of part (a) and part (b) of Lemma 2.6.1, but not part (c), hold for strictly supermodular functions and for functions with strictly increasing differences.

Lemma 2.6.1. *Suppose that X is a lattice.*
(a) *If $f(x)$ is supermodular on X and $\alpha > 0$, then $\alpha f(x)$ is supermodular on X.*
(b) *If $f(x)$ and $g(x)$ are supermodular on X, then $f(x) + g(x)$ is supermodular on X.*
(c) *If $f_k(x)$ is supermodular on X for $k = 1, 2, \dots$ and $\lim_{k \to \infty} f_k(x) = f(x)$ for each x in X, then $f(x)$ is supermodular on X.*

A useful consequence of Lemma 2.6.1, as formally stated in Corollary 2.6.2, is that the expected value of a supermodular function is supermodular. A similar property holds for concave functions. (In the statement of Corollary 2.6.2, assume that $g(x, w)$ is integrable with respect to $F(w)$ for each x in X.)

Corollary 2.6.2. *If $F(w)$ is a distribution function on a set W and $g(x, w)$ is supermodular in x on a lattice X for each w in W, then $\int_W g(x, w)dF(w)$ is supermodular in x on X.*

Example 2.6.3 gives another simple but useful application of Lemma 2.6.1.

Example 2.6.3. Consider the model of Example 2.1.2, where a consumer chooses a vector $x = (x_1, \dots, x_n)$ of consumption levels for n products from a subset X of R^n, the utility of a vector x of consumption levels is a real-valued function $f(x)$, and $p = (p_1, \dots, p_n)$ is the vector of unit prices for acquiring the n products. Suppose that X is a sublattice of R^n and $f(x)$ is supermodular on X, so the n products are complementary by Theorem 2.6.1. The net value for the consumer is $f(x) - p \cdot x$. Define $t = -p$, so the net value is $f(x) + t \cdot x$. By part (d) of Example 2.6.2 and part (b) of Lemma 2.6.1, the net value $f(x) + t \cdot x$ is supermodular in (x, t) on $X \times R^n$.

Lemma 2.6.2 gives necessary and sufficient conditions, under certain regularity conditions, for the composition $g(\sum_{i=1}^n a_i x_i)$ of a function $g(y)$ with a linear function $\sum_{i=1}^n a_i x_i$ to be supermodular in x. A necessary condition is that if $g(y)$ is not an affine function and at least two of the coefficients $a_1, \dots, a_n$ are nonzero then either all the nonzero coefficients of $a_1, \dots, a_n$ have the same sign or exactly two of the coefficients $a_1, \dots, a_n$ are nonzero and these two coefficients have different signs. Where all the nonzero coefficients of $a_1, \dots, a_n$ have the same sign and there are at least two nonzero coefficients, $g(\sum_{i=1}^n a_i x_i)$ is supermodular in x if and only if $g(y)$ is convex in y. Where there are exactly two nonzero coefficients among $a_1, \dots, a_n$ and these have different signs, $g(\sum_{i=1}^n a_i x_i)$ is supermodular in x if and only if $g(y)$ is concave in y.

Lemma 2.6.2. *Suppose that Y is a convex subset of the real line, X is a sublattice of R^n, a_i is a real number for $i = 1, \dots, n$, $\sum_{i=1}^n a_i x_i$ is in Y for each x in X, $g(y)$ is a real-valued function on Y, and $f(x) = g(\sum_{i=1}^n a_i x_i)$ for x in X.*
(a) *If $a_i > 0$ for $i = 1, \dots, n$ and $g(y)$ is convex in y on Y, then $f(x)$ is supermodular in x on X.*
(b) *If $n = 2$, $a_1 > 0$, $a_2 < 0$, and $g(y)$ is concave in y on Y, then $f(x)$ is supermodular in x on X.*

Now suppose, in addition, that X is an increasing set, $Y = \{\sum_{i=1}^n a_i x_i : x \in X\}$, and $g(y)$ is continuous on the interior of Y.

(c) *If* $n \geq 2$, $a_i > 0$ *for* $i = 1, \ldots, n$, *and* $f(x)$ *is supermodular in* x *on* X, *then* $g(y)$ *is convex in* y *on* Y.
(d) *If* $n = 2$, $a_1 > 0$, $a_2 < 0$, *and* $f(x)$ *is supermodular in* x *on* X, *then* $g(y)$ *is concave in* y *on* Y.
(e) *If* $n \geq 3$, $a_1 > 0$, $a_2 > 0$, $a_3 < 0$, *and* $f(x)$ *is supermodular in* x *on* X, *then* $g(y)$ *is an affine function in* y *on* Y.

Proof. Assume the hypotheses of part (a). Pick any x' and x'' in X. Let $\delta = \sum_{i=1}^n a_i(x'_i \vee x''_i - x'_i) = \sum_{i=1}^n a_i(x''_i - x'_i \wedge x''_i)$, $y' = \sum_{i=1}^n a_i(x'_i \wedge x''_i)$, and $y'' = \sum_{i=1}^n a_i x'_i$. Here, $\delta \geq 0$ and $y' \leq y''$. Then

$$\begin{aligned} &f(x' \vee x'') + f(x' \wedge x'') - f(x') - f(x'') \\ &\quad = g\big(\textstyle\sum_{i=1}^n a_i(x'_i \vee x''_i)\big) + g\big(\sum_{i=1}^n a_i(x'_i \wedge x''_i)\big) \\ &\qquad - g\big(\textstyle\sum_{i=1}^n a_i x'_i\big) - g\big(\sum_{i=1}^n a_i x''_i\big) \\ &\quad = (g(y'' + \delta) - g(y'')) - (g(y' + \delta) - g(y')) \\ &\quad \geq 0 \end{aligned}$$

because the convexity of $g(y)$ implies that $g(y + \delta) - g(y)$ is increasing in y for each $\delta > 0$. This establishes part (a).

Assume the hypotheses of part (b). Pick any unordered x' and x'' in X. Without loss of generality, suppose that $x'_1 < x''_1$ and $x''_2 < x'_2$. Let $\delta = a_2(x''_2 - x'_2)$, $y' = a_1 x'_1 + a_2 x'_2$, and $y'' = a_1 x''_1 + a_2 x'_2$. Here, $\delta > 0$ and $y' < y''$. Then

$$\begin{aligned} &f(x' \vee x'') + f(x' \wedge x'') - f(x') - f(x'') \\ &\quad = g(a_1(x'_1 \vee x''_1) + a_2(x'_2 \vee x''_2)) + g(a_1(x'_1 \wedge x''_1) + a_2(x'_2 \wedge x''_2)) \\ &\qquad - g(a_1 x'_1 + a_2 x'_2) - g(a_1 x''_1 + a_2 x''_2) \\ &\quad = g(a_1 x''_1 + a_2 x'_2) + g(a_1 x'_1 + a_2 x''_2) - g(a_1 x'_1 + a_2 x'_2) - g(a_1 x''_1 + a_2 x''_2) \\ &\quad = (g(y' + \delta) - g(y')) - (g(y'' + \delta) - g(y'')) \\ &\quad \geq 0 \end{aligned}$$

because the concavity of $g(y)$ implies that $g(y + \delta) - g(y)$ is decreasing in y for each $\delta > 0$. This establishes part (b).

Assume the hypotheses of part (c). Pick any y' and y'' in Y with $y' < y''$. Pick x' in X with $\sum_{i=1}^n a_i x'_i = y'$. Let $\epsilon_1 = (y'' - y')/2a_1 > 0$ and $\epsilon_2 =$

$(y'' - y')/2a_2 > 0$. Then

$$\begin{aligned}
&(1/2)g(y'') + (1/2)g(y') - g((1/2)y'' + (1/2)y') \\
&\quad = (1/2)(f(x' + \epsilon_1 u^1 + \epsilon_2 u^2) + f(x') - f(x' + \epsilon_1 u^1) - f(x' + \epsilon_2 u^2)) \\
&\quad = (1/2)(f((x' + \epsilon_1 u^1) \vee (x' + \epsilon_2 u^2)) + f((x' + \epsilon_1 u^1) \wedge (x' + \epsilon_2 u^2)) \\
&\qquad - f(x' + \epsilon_1 u^1) - f(x' + \epsilon_2 u^2)) \\
&\quad \geq 0
\end{aligned}$$

by the supermodularity of $f(x)$. Thus, using the continuity of $g(y)$ on the interior of Y, $g(y)$ is convex on Y. This establishes part (c).

Assume the hypotheses of part (d). Pick any y' and y'' in Y with $y' < y''$. Pick x' in X with $\sum_{i=1}^n a_i x_i' = (y' + y'')/2$. Let $\epsilon_1 = (y'' - y')/2a_1 > 0$ and $\epsilon_2 = -(y'' - y')/2a_2 > 0$. Then

$$\begin{aligned}
&g((1/2)y' + (1/2)y'') - (1/2)g(y') - (1/2)g(y'') \\
&\quad = (1/2)(f(x' + \epsilon_1 u^1 + \epsilon_2 u^2) + f(x') - f(x' + \epsilon_1 u^1) - f(x' + \epsilon_2 u^2)) \\
&\quad = (1/2)(f((x' + \epsilon_1 u^1) \vee (x' + \epsilon_2 u^2)) + f((x' + \epsilon_1 u^1) \wedge (x' + \epsilon_2 u^2)) \\
&\qquad - f(x' + \epsilon_1 u^1) - f(x' + \epsilon_2 u^2)) \\
&\quad \geq 0
\end{aligned}$$

by the supermodularity of $f(x)$. Thus, using the continuity of $g(y)$ on the interior of Y, $g(y)$ is concave on Y. This establishes part (d).

Under the hypotheses of part (e), part (c) and part (d) imply that $g(y)$ is both convex and concave and so $g(y)$ is an affine function on Y. □

Example 2.6.4 shows that assuming $g(y)$ to be continuous on the interior of its domain is crucial for the necessity results in part (c), part (d), and part (e) of Lemma 2.6.2. If $f(x)$ is a real-valued function on a set X for which the sum (+) of any two elements of X is also in X and if

$$f(x') + f(x'') \leq f(x' + x'')$$

for all x' and x'' in X, then $f(x)$ is **superadditive** on X. If $-f(x)$ is superadditive on X, then $f(x)$ is **subadditive** on X. If $f(x)$ is both superadditive and subadditive on X, then $f(x)$ is **additive** on X.

Example 2.6.4. By a well-known construction (Hardy, Littlewood, and Pólya [1934]), there exists a function on the real line that is additive and is nowhere continuous. Let $g(y)$ be such a function. Because a convex function is continuous on the relative interior of its domain (Rockafellar [1970]), $g(y)$ is neither convex nor concave nor affine on the real line. Let $a_1, \ldots, a_n$ be any real numbers. Pick any x' and x'' in R^n. The additivity of $g(y)$ implies that

$$\begin{aligned} g\big(\textstyle\sum_{i=1}^n a_i(x'_i \vee x''_i)\big) &- g\big(\textstyle\sum_{i=1}^n a_i x'_i\big) \\ &= g\big(\textstyle\sum_{i=1}^n a_i(x'_i \vee x''_i - x'_i)\big) \\ &= g\big(\textstyle\sum_{i=1}^n a_i(x''_i - x'_i \wedge x''_i)\big) \\ &= g\big(\textstyle\sum_{i=1}^n a_i x''_i\big) - g\big(\textstyle\sum_{i=1}^n a_i(x'_i \wedge x''_i)\big), \end{aligned}$$

so $g(\sum_{i=1}^n a_i x_i)$ is a valuation on R^n and hence is supermodular as well as submodular.

Consider a firm that produces n products. The production vector for the n products is a nonnegative n-vector x. The firm's cost function for a production vector x is $c(x)$. The firm's cost function exhibits **weak cost complementarity** if

$$c(x' + x'') - c(x') \geq c(x' + x'' + x''') - c(x' + x''') \tag{2.6.1}$$

for all nonnegative n-vectors x', x'', and x''' with $x''_i > 0$ implying $x'''_i = 0$ for $i = 1, \ldots, n$. (Then $x'' \vee x''' = x'' + x'''$ and $x'' \wedge x''' = 0$.) Weak cost complementarity is equivalent to the property that $c(x' + x'') - c(x') \geq c(x' + x'' + \epsilon u^i) - c(x' + \epsilon u^i)$ for all nonnegative n-vectors x' and x'', each i with $x''_i = 0$, and all $\epsilon > 0$, and this property is equivalent to $c(x)$ having decreasing differences in x for $x \geq 0$. By Theorem 2.6.1 and Corollary 2.6.1, weak cost complementarity is equivalent to the firm's cost function $c(x)$ being submodular in x for $x \geq 0$ as Sharkey [1982c] notes. If the cost function $c(x)$ is twice differentiable for $x \geq 0$, then weak cost complementarity is equivalent to $\partial^2 c(x)/\partial x_{i'} \partial x_{i''} \leq 0$ for all distinct products i' and i''. The firm's cost function exhibits **cost complementarity** if (2.6.1) holds for all nonnegative n-vectors x', x'', and x'''. Cost complementarity is equivalent to the property that $c(x + y)$ is submodular in (x, y) for $x \geq 0$ and $y \geq 0$. Hence, cost complementarity implies that $c(x)$ is submodular in x for $x \geq 0$ and, by part (c) of Lemma 2.6.2, cost complementarity and continuity imply that $c(x)$ is concave in each component x_i for $x \geq 0$. (By Example 2.6.4, cost complementarity without also assuming continuity does not imply that $c(x)$ is concave in each component x_i even for the case $n = 1$.) Conversely, if $c(x)$ is submodular in x and is concave in each component x_i, then $c(x + y)$ is submodular in (x_i, y_i) for each i by part (a) of Lemma 2.6.2 as well as being submodular in x and in y. Therefore, $c(x + y)$ has decreasing differences in each pair of distinct components from the $2n$-vector (x, y) by Theorem 2.6.1, and $c(x + y)$ is submodular in (x, y) by Corollary 2.6.1. If the cost function $c(x)$ is twice differentiable for

$x \geq 0$, then cost complementarity is equivalent to $\partial^2 c(x)/\partial x_{i'}\partial x_{i''} \leq 0$ for all (not necessarily distinct) products i' and i''. (In the terminology of the present monograph, the property of complementarity among the n products of the firm corresponds to weak cost complementarity for the firm's production cost function with cost complementarity being a stronger property.) The firm's cost function exhibits **economies of scope** if

$$c(x') + c(x'') \geq c(x' + x'')$$

for all pairs of nonnegative n-vectors x' and x'' with $x' \wedge x'' = 0$. The cost function $c(x)$ is **supportable** if, for each nonnegative production vector x' with $x' \neq 0$, there exists an n-vector p' of prices such that $\sum_{i=1}^{n} p'_i x'_i = c(x')$ and $\sum_{i=1}^{n} p'_i x''_i \leq c(x'')$ for each n-vector x'' with $0 \leq x'' \leq x'$; that is, with price vector p', the profit is 0 given the production vector x' and the profit is not positive for any production vector bounded above by x'. Sharkey and Telser [1978] consider necessary and sufficient conditions for a natural monopoly firm to have a supportable production cost function. If each firm in an industry has the production cost function $c(x)$ and if $c(x)$ is subadditive for $x \geq 0$, then the industry is a **natural monopoly** (Sharkey [1982c]). If $c(x)$ is subadditive, then the cost function $c(x)$ exhibits economies of scope. If $c(0) = 0$, then cost complementarity implies subadditivity (Sharkey [1982c]) and weak cost complementarity implies economies of scope.

Lemma 2.6.3, from Sharkey and Telser [1978], shows that cost complementarity together with several technical assumptions implies that a firm's production cost function is supportable. (The statement of Sharkey and Telser [1978] omits the hypothesis that $c(x)$ is continuous. Their proof incorrectly claims that continuity is implied by cost complementarity. As noted above with reference to Example 2.6.4, the property used in the following proof that $c(x)$ is concave in each component relies on a continuity hypothesis that must be made explicitly.)

Lemma 2.6.3. *Consider a firm producing nonnegative amounts of each of n products, where x in R^n is the production vector for the n products and $c(x)$ is the firm's cost function for a production vector $x \geq 0$. If $c(x)$ exhibits cost complementarity, $c(x)$ is continuous for $x \geq 0$, and $c(0) \geq 0$, then $c(x)$ is supportable.*

Proof. Pick any nonnegative production vector $x' \neq 0$. Let i' be the smallest integer i with $x'_i > 0$. Define the price vector p' such that $p'_i = 0$ for each i with $x'_i = 0$, $p'_{i'} = c(x'_1, \ldots, x'_{i'}, 0, \ldots, 0)/x'_{i'} = c(x'_{i'}u^{i'})/x'_{i'}$, and $p'_i = (c(x'_1, \ldots, x'_i, 0, \ldots, 0) - c(x'_1, \ldots, x'_{i-1}, 0, \ldots, 0))/x'_i$ for $i = i'+1, \ldots, n$ with $x'_i > 0$.

Using the definitions of p' and i',

$$\begin{aligned}\textstyle\sum_{i=1}^n p'_i x'_i &= p'_{i'} x'_{i'} + \textstyle\sum_{\{i:i>i',x'_i>0\}} p'_i x'_i \\ &= c(x'_1,\ldots,x'_{i'},0,\ldots,0) \\ &\quad + \textstyle\sum_{\{i:i>i',x'_i>0\}}(c(x'_1,\ldots,x'_i,0,\ldots,0) \\ &\quad - c(x'_1,\ldots,x'_{i-1},0,\ldots,0)) \\ &= c(x'_1,\ldots,x'_{i'},0,\ldots,0) \\ &\quad + \textstyle\sum_{i=i'+1}^n(c(x'_1,\ldots,x'_i,0,\ldots,0) - c(x'_1,\ldots,x'_{i-1},0,\ldots,0)) \\ &= c(x').\end{aligned}$$

Now pick any n-vector x'' with $0 \le x'' \le x'$. For any $i > i'$ with $x'_i > 0$,

$$\begin{aligned}p'_i(x'_i - x''_i) &= (c(x'_1,\ldots,x'_i,0,\ldots,0) - c(x'_1,\ldots,x'_{i-1},0,\ldots,0))(x'_i - x''_i)/x'_i \\ &\ge (c(x'_1,\ldots,x'_i,x''_{i+1},\ldots,x''_n) \\ &\quad - c(x'_1,\ldots,x'_{i-1},0,x''_{i+1},\ldots,x''_n))(x'_i - x''_i)/x'_i \\ &\ge c(x'_1,\ldots,x'_i,x''_{i+1},\ldots,x''_n) - c(x'_1,\ldots,x'_{i-1},x''_i,\ldots,x''_n)\end{aligned}$$

where the first inequality follows from the submodularity of $c(x)$ in x as implied by cost complementarity and the second inequality follows from the concavity of $c(x)$ in x_i as implied by cost complementarity and continuity. Likewise, using $c(0) \ge 0$,

$$\begin{aligned}p'_{i'}(x'_{i'} - x''_{i'}) &= c(x'_1,\ldots,x'_{i'},0,\ldots,0)(x'_{i'} - x''_{i'})/x'_{i'} \\ &\ge (c(x'_1,\ldots,x'_{i'},0,\ldots,0) - c(0))(x'_{i'} - x''_{i'})/x'_{i'} \\ &\ge (c(x'_1,\ldots,x'_{i'},x''_{i'+1},\ldots,x''_n) \\ &\quad - c(x'_1,\ldots,x'_{i'-1},0,x''_{i'+1},\ldots,x''_n))(x'_{i'} - x''_{i'})/x'_{i'} \\ &\ge c(x'_1,\ldots,x'_{i'},x''_{i'+1},\ldots,x''_n) - c(x'_1,\ldots,x'_{i'-1},x''_{i'},\ldots,x''_n).\end{aligned}$$

Using the preceding equality and two inequalities,

$$\begin{aligned}\textstyle\sum_{i=1}^n p'_i x''_i &\le \textstyle\sum_{i=1}^n p'_i x'_i + \textstyle\sum_{i=1}^n(c(x'_1,\ldots,x'_{i-1},x''_i,\ldots,x''_n) \\ &\quad - c(x'_1,\ldots,x'_i,x''_{i+1},\ldots,x''_n)) \\ &= c(x') + c(x'') - c(x') = c(x''). \quad \square\end{aligned}$$

Lemma 2.6.4 shows how one may construct a supermodular function by taking a generalized composition of other functions having certain properties. The special case of Lemma 2.6.4 for $k = 1$ is from Topkis [1995a]. The special case of Lemma 2.6.4 (with $k = 1$ and $g(f_1(x), x) = g(f_1(x))$) that an increasing (decreasing) convex function of an increasing supermodular (submodular) function is supermodular is from Topkis [1978]. The latter special case corresponds to the composition result that an increasing convex (concave) function of a convex (concave) function is convex (concave) (Rockafellar [1970]).

Lemma 2.6.4. *If X is a lattice, $f_i(x)$ is increasing and supermodular (submodular) on X for $i = 1, \ldots, k$, Z_i is a convex subset of R^1 containing the range of $f_i(x)$ on X for $i = 1, \ldots, k$, and $g(z_1, \ldots, z_k, x)$ is supermodular in $(z_1, \ldots, z_k, x)$ on $(\times_{i=1}^k Z_i) \times X$ and is increasing (decreasing) and convex in z_i on Z_i for $i = 1, \ldots, k$ and for all $z_{i'}$ in $Z_{i'}$ for i' in $\{1, \ldots, n\} \setminus \{i\}$ and all x in X, then $g(f_1(x), \ldots, f_k(x), x)$ is supermodular on X.*

Proof. Pick any x' and x'' in X. Note that

$$
\begin{aligned}
& g(f_1(x' \vee x''), \ldots, f_k(x' \vee x''), x' \vee x'') \\
& \quad - g(f_1(x' \vee x'') + f_1(x' \wedge x'') - f_1(x''), \ldots, \\
& \qquad f_k(x' \vee x'') + f_k(x' \wedge x'') - f_k(x''), x' \vee x'') \\
& = \textstyle\sum_{i=1}^k (g(f_1(x' \vee x''), \ldots, f_i(x' \vee x''), \\
& \qquad f_{i+1}(x' \vee x'') + f_{i+1}(x' \wedge x'') - f_{i+1}(x''), \ldots, \\
& \qquad f_k(x' \vee x'') + f_k(x' \wedge x'') - f_k(x''), x' \vee x'') \\
& \quad - g(f_1(x' \vee x''), \ldots, f_{i-1}(x' \vee x''), \\
& \qquad f_i(x' \vee x'') + f_i(x' \wedge x'') - f_i(x''), \ldots, \\
& \qquad f_k(x' \vee x'') + f_k(x' \wedge x'') - f_k(x''), x' \vee x'')) \\
& \geq \textstyle\sum_{i=1}^k (g(f_1(x''), \ldots, f_{i-1}(x''), f_i(x' \vee x''), \qquad (2.6.2) \\
& \qquad f_{i+1}(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x'') \\
& \quad - g(f_1(x''), \ldots, f_{i-1}(x''), f_i(x' \vee x'') + f_i(x' \wedge x'') - f_i(x''), \\
& \qquad f_{i+1}(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x'')) \\
& \geq \textstyle\sum_{i=1}^k (g(f_1(x''), \ldots, f_{i-1}(x''), f_i(x''), \\
& \qquad f_{i+1}(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x'') \\
& \quad - g(f_1(x''), \ldots, f_{i-1}(x''), f_i(x' \wedge x''), \\
& \qquad f_{i+1}(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x'')) \\
& = g(f_1(x''), \ldots, f_k(x''), x' \vee x'') \\
& \quad - g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x''),
\end{aligned}
$$

where the two equalities are identities, the first inequality follows because each $f_i(x)$ is increasing and $g(z_1, \ldots, z_k, x)$ is supermodular, and the second inequality follows because each $f_i(x)$ is increasing and $g(z_1, \ldots, z_k, x)$ is convex in each z_i.

Then,

$$\begin{aligned}
&g(f_1(x' \vee x''), \ldots, f_k(x' \vee x''), x' \vee x'')\\
&\qquad + g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \wedge x'')\\
&\qquad - g(f_1(x'), \ldots, f_k(x'), x') - g(f_1(x''), \ldots, f_k(x''), x'')\\
&\quad = g(f_1(x' \vee x''), \ldots, f_k(x' \vee x''), x' \vee x'')\\
&\qquad - g(f_1(x' \vee x'') + f_1(x' \wedge x'') - f_1(x''), \ldots,\\
&\qquad\qquad f_k(x' \vee x'') + f_k(x' \wedge x'') - f_k(x''), x' \vee x'')\\
&\qquad - g(f_1(x''), \ldots, f_k(x''), x' \vee x'')\\
&\qquad + g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x'')\\
&\qquad + g(f_1(x' \vee x'') + f_1(x' \wedge x'') - f_1(x''), \ldots,\\
&\qquad\qquad f_k(x' \vee x'') + f_k(x' \wedge x'') - f_k(x''), x' \vee x'')\\
&\qquad - g(f_1(x'), \ldots, f_k(x'), x' \vee x'')\\
&\qquad + g(f_1(x''), \ldots, f_k(x''), x' \vee x'') + g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x'')\\
&\qquad - g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \vee x'') - g(f_1(x''), \ldots, f_k(x''), x'')\\
&\qquad + g(f_1(x'), \ldots, f_k(x'), x' \vee x'')\\
&\qquad + g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x' \wedge x'')\\
&\qquad - g(f_1(x' \wedge x''), \ldots, f_k(x' \wedge x''), x'') - g(f_1(x'), \ldots, f_k(x'), x')\\
&\quad \geq 0,
\end{aligned}$$

where the equality is an identity, the sum of the terms in the first five lines after the equality is nonnegative by (2.6.2), the sum of the terms in the sixth through eighth lines after the equality is nonnegative because each $f_i(x)$ is supermodular (submodular) in x and $g(z_1, \ldots, z_k, x)$ is increasing (decreasing) in each z_i, and the sum of the terms in the ninth and tenth lines after the equality and the sum of the terms in the eleventh through thirteenth lines after the equality are both nonnegative because each $f_i(x)$ is increasing in x and $g(z_1, \ldots, z_k, x)$ is supermodular in $(z_1, \ldots, z_k, x)$. □

Corollary 2.6.3, from Topkis [1978], shows that the product of nonnegative increasing (decreasing) supermodular functions is supermodular.

Corollary 2.6.3. *If X is a lattice and $f_i(x)$ is nonnegative, increasing (decreasing), and supermodular on X for $i = 1, \ldots, k$, then $f_1(x)f_2(x)\cdots f_k(x)$ is also nonnegative, increasing (decreasing), and supermodular on X.*

Proof. That $f_1(x)f_2(x)\cdots f_k(x)$ is nonnegative and increasing (decreasing) is immediate. The supermodularity of $f_1(x)f_2(x)\cdots f_k(x)$ follows from Lemma 2.6.4 by letting $g(z_1, \ldots, z_k, x) = z_1 z_2 \ldots z_k$. (For the case of this result in parentheses, the supermodularity of $f_1(x)f_2(x)\cdots f_k(x)$ follows from Lemma 2.6.4 by letting $-f_i(x)$ correspond to the $f_i(x)$ in the statement of Lemma 2.6.4 for $i = 1, \ldots, k$, letting $g(z_1, \ldots, z_k, x) = z_1 z_2 \ldots z_k$ if k is even, and letting $g(z_1, \ldots, z_k, x) = -z_1 z_2 \ldots z_k$ if k is odd.) □

Example 2.6.5 shows that a strictly increasing transformation of a supermodular function need not be supermodular. This failure to be preserved under a strictly increasing transformation is a source of some criticism for using the cardinal property of supermodular utility to represent complementarity.

Example 2.6.5. Let $X = [1, \infty) \times [1, \infty)$, $f(x) = x_1 + x_2$ for $x = (x_1, x_2)$ in X, $g(y) = \log(y)$ for $y > 0$, and $h(x) = g(f(x))$ for x in X. Then $\partial^2 h(x)/\partial x_1 \partial x_2 = -1/(x_1 + x_2)^2 < 0$. Thus $h(x)$ is not supermodular on X by Theorem 2.6.1 even though it is a strictly increasing function of a strictly increasing supermodular function. Indeed, by Corollary 2.6.1, $h(x)$ is strictly submodular on X.

2.6.3 Ordinal Generalizations

The use of the notion of increasing differences (equivalently, supermodularity) to express the concept of complementarity is subject to criticism (Samuelson [1947]) as being a cardinal property that depends on numerical utility values and, as in Example 2.6.5, may not be preserved by a strictly increasing transformation. There are several responses to this valid observation. One response is that, by Lemma 2.6.4, an increasing transformation does indeed preserve the supermodularity of an increasing supermodular function if that transformation also exhibits increasing returns to scale (that is, is convex). A second response, as developed largely in Section 2.8 and summarized in Section 2.9, is that supermodularity is equivalent to several other common notions for complementarity and that it implies another property, that $\text{argmax}_{x \in S_t} f(x, t)$ is increasing in t, which is intimately related to complementarity and which a strictly increasing transformation preserves. A third response is that, notwithstanding its cardinal limitations, supermodularity provides a powerful, attractive, and useful set of tools that are applicable in a wide variety of economic problems. A fourth response is to look beyond supermodularity and to instead focus on whatever properties are essential for a utility function to imply important qualitative properties such as monotone comparative statics. Reflecting the latter perspective, Milgrom and Shannon [1994] introduce and study quasisupermodular functions and functions with the single crossing property.

If $f(x)$ is a function from a lattice X into a partially ordered set Y, then $f(x)$ is **quasisupermodular** if, for all x' and x'' in X, $f(x' \wedge x'') \preceq f(x')$ implies $f(x'') \preceq f(x' \vee x'')$ and $f(x' \wedge x'') \prec f(x')$ implies $f(x'') \prec f(x' \vee x'')$. If X, T, and Y are partially ordered sets and $f(x, t)$ is a function from a subset S of $X \times T$ into Y, then $f(x, t)$ satisfies the **single crossing property** in (x, t) on S if, for all x' and x'' in X and t' and t'' in T with $x' \prec x''$, $t' \prec t''$, and $\{x', x''\} \times \{t', t''\}$ being a subset of S, $f(x', t') \preceq f(x'', t')$ implies $f(x', t'') \preceq f(x'', t'')$ and $f(x', t') \prec f(x'', t')$ implies $f(x', t'') \prec f(x'', t'')$. If X, T, and Y are partially ordered sets and $f(x, t)$ is a function from a subset S of $X \times T$ into Y, then $f(x, t)$ satisfies the **strict single crossing property** in (x, t) on S if, for all x' and x'' in X and t' and t'' in T with $x' \prec x''$, $t' \prec t''$, and $\{x', x''\} \times \{t', t''\}$ being a subset of S, $f(x', t') \preceq f(x'', t')$ implies $f(x', t'') \prec f(x'', t'')$. (Milgrom and Shannon [1994] define quasisupermodular functions and functions satisfying the single crossing property as being real-valued, but that restriction is not relied upon in their analysis and, in the spirit of viewing these conditions as ordinal, the ranges are here taken to be arbitrary partially ordered sets.) The single crossing property corresponds to an ordinal version of complementarity between x and t with respect to an ordinal utility function $f(x, t)$, in the sense that if x'' is (strictly) preferred to x' given $t = t'$ where $x' \prec x''$ then x'' is (strictly) preferred to x' given $t = t''$ where $t' \prec t''$; that is, if it is (strictly) preferable to have more of the first component given a particular level for the second component, then it would still be (strictly) preferable to have more of the first component given a greater level for the second component. However, unlike with increasing differences, there is no sense of a comparatively greater additional preference for more of the first component resulting from a higher level of the second component.

Lemma 2.6.5, from Milgrom and Shannon [1994], summarizes some basic properties of quasisupermodular functions and functions with the single crossing property. These functions generalize supermodular functions and functions with increasing differences (part (b) and part (e) of Lemma 2.6.5). Some properties of quasisupermodular functions correspond to similar properties of supermodular functions. Quasisupermodularity on the direct product of two lattices implies that the single crossing property holds on that product set (part (c) of Lemma 2.6.5). (See Theorem 2.6.1.) The single crossing property on the direct product of two chains implies quasisupermodularity on that product set (part (d) of Lemma 2.6.5). (See Corollary 2.6.1 with $n = 2$.) Consequently, if

X is the direct product of n chains indexed $i = 1, \ldots, n$, elements of X are denoted $x = (x_1, \ldots, x_n)$ where x_i is in chain i, and $f(x_I, x_{N\setminus I})$ satisfies the single crossing property in $(x_I, x_{N\setminus I})$ for each subset I of $N = \{1, \ldots, n\}$ with $1 \leq |I| \leq n-1$, then $f(x)$ is quasisupermodular on X. Any function from a chain to a partially ordered set is quasisupermodular (part (a) of Lemma 2.6.5). (See part (a) of Example 2.6.2.) A quasisupermodular function from a lattice to a chain is also quasisupermodular on the dual of the lattice. A significant property for ordinal interpretations of quasisupermodular functions and functions with the single crossing property is that, for functions whose range is a chain, these properties are preserved under a strictly increasing transformation (part (f) and part (g) of Lemma 2.6.5). The proofs for Lemma 2.6.5 are direct, and so are omitted.

Lemma 2.6.5. *Suppose that $f(x)$ is a function from a set X into a partially ordered set Y.*
(a) *If X is a chain, then $f(x)$ is quasisupermodular on X.*
(b) *If X is a lattice, $Y = R^1$, and $f(x)$ is supermodular on X, then $f(x)$ is quasisupermodular on X.*
(c) *If X_1 and X_2 are lattices, X is a sublattice of $X_1 \times X_2$, and $f(x)$ is quasisupermodular on X, then $f(x_1, x_2)$ has the single crossing property in (x_1, x_2) and in (x_2, x_1) on X.*
(d) *If X_1 and X_2 are chains, X is a sublattice of $X_1 \times X_2$, and $f(x_1, x_2)$ has the single crossing property in (x_1, x_2) and in (x_2, x_1) on X, then $f(x)$ is quasisupermodular on X.*
(e) *If X_1 and X_2 are partially ordered sets, X is a subset of $X_1 \times X_2$, $Y = R^1$, and $f(x_1, x_2)$ has increasing differences in (x_1, x_2) on X, then $f(x_1, x_2)$ has the single crossing property in (x_1, x_2) and in (x_2, x_1) on X.*
(f) *If X is a lattice, Y is a chain, $f(x)$ is quasisupermodular on X, and $g(y)$ is a strictly increasing function from Y into a partially ordered set W, then $g(f(x))$ is a quasisupermodular function from X into W.*
(g) *If X_1 and X_2 are partially ordered sets, X is a subset of $X_1 \times X_2$, Y is a chain, $f(x_1, x_2)$ has the single crossing property in (x_1, x_2) on X, and $g(y)$ is a strictly increasing function from Y into a partially ordered set W, then $g(f(x))$ has the single crossing property in (x_1, x_2) on X.*

Any strictly increasing function of a supermodular function is quasisupermodular (from part (b) and part (f) of Lemma 2.6.5). However, the converse statement is not true, as Example 2.6.6, from Milgrom and Shannon [1994], exhibits a quasisupermodular function that is not a strictly increasing transformation of any supermodular function. (See Corollary 2.8.5, which limits the impact of the generality suggested by Example 2.6.6 for quasisupermodular functions beyond strictly increasing functions of a supermodular function.)

Example 2.6.6. Let $Y = \{a, b, c, d, e\}$ be a chain with $a \prec b \prec c \prec d \prec e$, and let $f(x)$ be the quasisupermodular function from the sublattice $\{1, 2\} \times \{1, 2, 3, 4\}$ of R^2 into Y where $f(1, 1) = f(1, 4) = a$, $f(1, 2) = f(1, 3) = b$, $f(2, 1) = f(2, 4) = c$, $f(2, 2) = d$, and $f(2, 3) = e$. Suppose that $g(y)$ is a strictly increasing function from Y into R^1 such that $g(f(x))$ is supermodular on $\{1, 2\} \times \{1, 2, 3, 4\}$. But this is impossible because then

$$\begin{aligned} g(c) - g(a) &= g(f(2,1)) - g(f(1,1)) \le g(f(2,2)) - g(f(1,2)) \\ &= g(d) - g(b) < g(e) - g(b) = g(f(2,3)) - g(f(1,3)) \\ &\le g(f(2,4)) - g(f(1,4)) = g(c) - g(a), \end{aligned}$$

where the two weak inequalities follow from the supermodularity of $g(f(x))$ and the strict inequality follows from $g(y)$ being strictly increasing. Therefore, there does not exist any strictly increasing real-valued function $g(y)$ on Y such that $g(f(x))$ is supermodular on $\{1, 2\} \times \{1, 2, 3, 4\}$. Consequently, $f(x)$ is not a strictly increasing transformation of a supermodular function.

The quasisupermodular function in Example 2.6.6 could have been represented equivalently (that is, by a strictly increasing transformation) as a real-valued function, because the range of that function is a finite chain. Example 2.6.7 exhibits a quasisupermodular function that is not a strictly increasing transformation of any real-valued function, much less a strictly increasing transformation of a supermodular function. The function of Example 2.6.7 is presented in Debreu [1959] as an instance of a strictly increasing function into a chain such that there is no strictly increasing transformation of the function into a real-valued function. That function is thus relevant for present purposes because any strictly increasing function (on a lattice) is quasisupermodular.

Example 2.6.7. Let $X = R^2$ and $f(x) = x$ for each x in X. The domain X of $f(x)$ has the usual ordering relation $\le$ on R^2, but the range of $f(x)$ for x in X (which is also the set of points X) has the lexicographic ordering relation $\le_{lex}$; that is, $f(x') \le_{lex} f(x'')$ either if $x'_1 < x''_1$ or if $x'_1 = x''_1$ and $x'_2 \le x''_2$. Then $f(x)$ is a strictly increasing (and hence quasisupermodular) function on the lattice $X = R^2$, and the range of $f(x)$ on X is a chain. Suppose that $g(x)$ is a real-valued function on X, $h(z)$ is a strictly increasing function from the range of $g(x)$ into X with the $\le_{lex}$ ordering relation, and $h(g(x)) = f(x) = x$ for each x in X. Then $g(x)$ is a strictly increasing function from X into R^1 where the ordering relation on X is $\le_{lex}$. For each z in R^1, define

$a(z) = \inf_{w \in R^1} g(z, w)$ and $b(z) = \sup_{w \in R^1} g(z, w)$. For each z in R^1, $a(z)$ and $b(z)$ are real-valued because $g(z-1, 0) \leq a(z)$ and $b(z) \leq g(z+1, 0)$. For each z in R^1, $a(z) < b(z)$ because $a(z) < g(z, 0) < b(z)$ and so one can pick a rational number $c(z)$ with $a(z) \leq c(z) \leq b(z)$. For each z' and z'' in R^1 with $z' < z''$, $b(z') < a(z'')$ because $b(z') < g((z' + z'')/2, 0) < a(z'')$. Therefore, $c(z)$ is strictly increasing. But this means that $c(z)$ is a one-to-one function from the uncountable set R^1 to a subset of the countable set of rationals in R^1, which is a contradiction.

The generalized characteristics, quasisupermodularity and the single crossing property, for a utility function offer several significant advantages. Quasisupermodular functions and functions with the single crossing property are ordinal conditions, since these conditions are preserved under strictly increasing transformations (part (f) and part (g) of Lemma 2.6.5). These properties enable generalizations of several key results for supermodular functions where $\mathrm{argmax}_{x \in X} f(x)$ is a sublattice (Theorem 2.7.2 generalizing Theorem 2.7.1) and where $\mathrm{argmax}_{x \in S_t} f(x, t)$ is increasing in t (Theorem 2.8.6 and Theorem 2.8.7 generalizing Theorem 2.8.1 and Theorem 2.8.4, respectively). They are a basis of necessary conditions for $\mathrm{argmax}_{x \in S_t} f(x, t)$ to be increasing in t (Theorem 2.8.13 and Theorem 2.8.14).

Some handy properties of supermodular functions and functions with increasing differences do not generally hold for quasisupermodular functions and functions satisfying the single crossing property, as the next three examples demonstrate. As in an example of Shannon [1995], Example 2.6.8 shows that, unlike the result of Corollary 2.6.1 for supermodular functions, on the direct product of more than two chains the single crossing property (equivalently, quasisupermodularity) in each pair of variables does not imply quasisupermodularity in all variables. Example 2.6.9 shows that the single crossing property is not symmetric in the two variables (as increasing differences is); that is, $f(x, t)$ may have the single crossing property in (x, t) but not in (t, x). Example 2.6.10 shows that a function that is the pointwise limit of a sequence of real-valued quasisupermodular functions (functions satisfying the single crossing property) need not be quasisupermodular (satisfy the single crossing property); part (c) of Lemma 2.6.1 notes that the pointwise limit of supermodular functions is supermodular. The sum of two real-valued quasisupermodular functions need not be quasisupermodular if either is not supermodular, and, even stronger, the sum of a real-valued quasisupermodular function and a linear function need not be quasisupermodular if the former is not supermodular (Theorem 2.6.5); part (b) of Lemma 2.6.1 notes that the sum of two supermodular functions is supermodular.

Example 2.6.8. Let $X_i = \{0, 1\}$ for $i = 1, 2, 3$, $X = \times_{i=1}^{3} X_i$, $f(0, 0, 0) = 3$, $f(0, 0, 1) = f(0, 1, 0) = f(1, 0, 0) = 2$, $f(0, 1, 1) = 4$, $f(1, 0, 1) = f(1, 1, 0) = 0$, and $f(1, 1, 1) = 1$. It can be seen by inspection that $f(x)$ is quasisupermodular and has the single crossing property in each pair of components of $x = (x_1, x_2, x_3)$. However, $f(x)$ is not quasisupermodular in $x = (x_1, x_2, x_3)$ because $f((1, 0, 0) \wedge (0, 1, 1)) = f(0, 0, 0) = 3 < 4 = f(0, 1, 1)$ and $f(1, 0, 0) = 2 > 1 = f(1, 1, 1) = f((1, 0, 0) \vee (0, 1, 1))$.

Example 2.6.9. Let $X = T = \{0, 1\}$ and define $f(x, t)$ on $X \times T$ such that $f(0, 0) = 1$, $f(1, 0) = 4$, $f(0, 1) = 2$, and $f(1, 1) = 3$. Then $f(x, t)$ satisfies the single crossing property in (x, t) but not in (t, x).

Example 2.6.10. Let $X_i = \{0, 1\}$ for $i = 1, 2$, $X = \times_{i=1}^{2} X_i$, and define $f_k(x)$ on X for $k = 1, 2, \ldots$ such that $f_k(0, 0) = 0$, $f_k(0, 1) = f_k(1, 0) = 1$, and $f_k(1, 1) = 1 + 1/k$. Let $f(x) = \lim_{k \to \infty} f_k(x)$ for each x in X. Each $f_k(x)$ is quasisupermodular on X, but $f(x)$ is not quasisupermodular on X because $f(0, 0) = 0 < 1 = f(0, 1)$ and $f(1, 0) = 1 \not< 1 = f(1, 1)$.

Theorem 2.6.5 and Theorem 2.6.6, from Milgrom and Shannon [1994], characterize supermodular functions and functions with increasing differences as the only real-valued functions on a sublattice of R^n that, respectively, are quasisupermodular and have the single crossing property after being perturbed by an arbitrary linear function.

Theorem 2.6.5. *If X is a sublattice of R^n and $f(x)$ is a real-valued function on X, then $f(x) + p \cdot x$ is quasisupermodular in x on X for each p in R^n if and only if $f(x)$ is supermodular in x on X.*

Proof. Suppose that $f(x)$ is supermodular in x on X and p is in R^n. By Theorem 2.6.4 and part (b) of Lemma 2.6.1, $f(x) + p \cdot x$ is supermodular in x on X. (See also Example 2.6.3 and part (d) of Example 2.6.2.) By part (b) of Lemma 2.6.5, $f(x) + p \cdot x$ is quasisupermodular in x on X.

Now suppose that $f(x)$ is not supermodular in x on X, so there exist x' and x'' in X with

$$f(x' \vee x'') + f(x' \wedge x'') < f(x') + f(x'').$$

Pick p' in R^n with

$$f(x' \vee x'') - f(x'') < p' \cdot (x'' - x' \vee x'') = p' \cdot (x' \wedge x'' - x') < f(x') - f(x' \wedge x'').$$

Then

$$f(x' \wedge x'') + p' \cdot (x' \wedge x'') < f(x') + p' \cdot x'$$

and

$$f(x' \vee x'') + p' \cdot (x' \vee x'') < f(x'') + p' \cdot x'',$$

and so $f(x) + p' \cdot x$ is not quasisupermodular in x on X. □

Theorem 2.6.6. *If T is a chain, S is a sublattice of $R^n \times T$, and $f(x, t)$ is a real-valued function on S with x in R^n and t in T, then $f(x, t) + p \cdot x$ has the single crossing property in (x, t) on S for each p in R^n if and only if $f(x, t)$ has increasing differences in (x, t) on S.*

Proof. Sufficiency follows from part (e) of Lemma 2.6.5.

Suppose that $f(x, t)$ does not have increasing differences in (x, t) on S, so there exist (x', t'') and (x'', t') in S with $x' < x''$, $t' \preceq t''$, and

$$f(x'', t'') - f(x', t'') < f(x'', t') - f(x', t').$$

Pick p' in R^n with

$$f(x'', t'') - f(x', t'') < p' \cdot (x' - x'') < f(x'', t') - f(x', t').$$

Then

$$f(x', t') + p' \cdot x' < f(x'', t') + p' \cdot x''$$

and

$$f(x'', t'') + p' \cdot x'' < f(x', t'') + p' \cdot x',$$

and so $f(x, t) + p' \cdot x$ does not have the single crossing property in (x, t) on S. □

2.6.4 Log-Supermodularity

If X is a lattice, $f(x)$ is a positive real-valued function on X, and $\log(f(x))$ is supermodular (submodular) on X, then $f(x)$ is **log-supermodular** (**log-submodular**). A positive real-valued function $f(x)$ on a lattice X is log-supermodular if and only if

$$f(x')f(x'') \le f(x' \vee x'')f(x' \wedge x'') \tag{2.6.3}$$

for all x' and x'' in X. Milgrom and Roberts [1990a] and Milgrom and Shannon [1994] give applications of log-supermodularity and the use of the log transformation. Because $f(x) = \exp(\log(f(x)))$, a log-supermodular function is quasisupermodular by part (b) and part (f) of Lemma 2.6.5. Any positive real-valued function on a chain is log-supermodular and log-submodular. By (2.6.3), the product of log-supermodular functions is log-supermodular. A positive scalar times a log-supermodular function is log-supermodular. The limit of a sequence of log-supermodular functions is log-supermodular if it is positive. Example 2.6.5 shows that the sum of two increasing log-supermodular functions need not be log-supermodular, and, indeed, a strictly increasing separable function is not generally log-supermodular. A real-valued function $f(x)$ on a product set $\times_{i=1}^{n} X_i$ is **multiplicatively separable** if $f(x) = \prod_{i=1}^{n} f_i(x_i)$ for all $x = (x_1, \dots, x_n)$ with x_i in X_i for $i = 1, \dots, n$. The Cobb-Douglas function (see part (e) of Example 2.6.2) is multiplicatively separable. By Theorem 2.6.4, a positive real-valued function on the direct product of a finite collection of chains is both log-supermodular and log-submodular if and only if it is multiplicatively separable. Lemma 2.6.6 establishes implications between supermodularity (submodularity) and log-supermodularity (log-submodularity). Example 2.6.11 relates monotonicity of the price elasticity of demand with log-supermodularity.

Lemma 2.6.6. *Suppose that $f(x)$ is a positive real-valued function on a lattice X.*
(a) *If $f(x)$ is increasing (decreasing) and log-supermodular, then $f(x)$ is supermodular.*
(b) *If $f(x)$ is increasing (decreasing) and submodular, then $f(x)$ is log-submodular.*

Proof. Pick any x' and x'' in X.

If $f(x)$ is increasing (decreasing) and log-supermodular, then

$$\begin{aligned} f(x')/f(x' \wedge x'') - 1 &\leq f(x' \vee x'')/f(x'') - 1 \\ &\leq (f(x' \vee x'')/f(x'') - 1)(f(x'')/f(x' \wedge x'')). \end{aligned}$$

Therefore,

$$f(x') + f(x'') \leq f(x' \vee x'') + f(x' \wedge x'')$$

and $f(x)$ is supermodular.

If $f(x)$ is increasing (decreasing) and submodular, then

$$\begin{aligned} f(x' \vee x'')/f(x'') - 1 &\leq (f(x')/f(x' \wedge x'') - 1)(f(x' \wedge x'')/f(x'')) \\ &\leq f(x')/f(x' \wedge x'') - 1. \end{aligned}$$

Therefore,

$$f(x' \vee x'')f(x' \wedge x'') \leq f(x')f(x'')$$

and $f(x)$ is log-submodular. □

Example 2.6.11. Suppose that $f(p, t)$ is the demand function for a product with price p and parameter t, where $f(p, t)$ is positive and differentiable in p

and t is contained in some partially ordered set. The price elasticity of demand is $-(\partial f(p,t)/\partial p)/(f(p,t)/p) = -(p)(\partial(\log f(p,t))/\partial p)$, which is decreasing in t if and only if $\log f(p,t)$ has increasing differences in (p,t) as Milgrom and Roberts [1990a] note. Therefore, if the demand function $f(p,t)$ is log-supermodular in (p,t) then the price elasticity of demand is decreasing in t by Theorem 2.6.1, and if the price elasticity of demand is decreasing in t and t is constrained to be in a chain then the demand function $f(p,t)$ is log-supermodular in (p,t) by Corollary 2.6.1.

2.7 Maximizing a Supermodular Function

This section examines the problem of maximizing a supermodular function on a lattice. Subsection 2.7.1 develops properties involving sets of maximizers. Subsection 2.7.2 considers the preservation of supermodularity under the maximization operation.

2.7.1 Sets of Maximizers

Theorem 2.7.1, from Topkis [1978], shows that the set of points at which a supermodular function attains its maximum on a lattice is a sublattice. Ore [1962] gives a general proof of this result in the context of finding a maximal deficiency in a network. Theorem 2.7.1 is similar in form to the property that the set of maxima of a concave function on a convex set is a convex set (Rockafellar [1970]).

Theorem 2.7.1. *If $f(x)$ is supermodular on a lattice X, then* $\text{argmax}_{x \in X} f(x)$ *is a sublattice of X.*

Proof. Pick any x' and x'' in $\text{argmax}_{x \in X} f(x)$. Because $f(x)$ is supermodular on the lattice X and x' and x'' are in $\text{argmax}_{x \in X} f(x)$,

$$0 \le f(x'') - f(x' \wedge x'') \le f(x' \vee x'') - f(x') \le 0$$

and so $x' \vee x''$ and $x' \wedge x''$ are in $\text{argmax}_{x \in X} f(x)$. □

In considering subsequent optimization problems, it is useful to have a consistent method of selecting one element from the set of optimal solutions. One way is to pick that set's greatest element or least element if these exist. Corollary 2.7.1 provides conditions for the set of optimal solutions to have a greatest element and a least element. Part (a) of Theorem 2.8.3 applies this result to select a particular optimal solution corresponding to each parameter value in a parameterized collection of optimization problems.

Corollary 2.7.1. *Suppose that $f(x)$ is supermodular on a nonempty lattice X.*
(a) *If X is finite, then* $\operatorname{argmax}_{x\in X} f(x)$ *is a nonempty subcomplete sublattice of X.*
(b) *If X is a compact sublattice of R^n and f(x) is upper semicontinuous on X, then* $\operatorname{argmax}_{x\in X} f(x)$ *is a nonempty compact and subcomplete sublattice of R^n. Hence, under the conditions of either* (a) *or* (b), $\operatorname{argmax}_{x\in X} f(x)$ *has a greatest element and a least element.*

Proof. Suppose that X is finite, so $\operatorname{argmax}_{x\in X} f(x)$ is nonempty. By Theorem 2.7.1, $\operatorname{argmax}_{x\in X} f(x)$ is a sublattice of X. By Lemma 2.2.1, $\operatorname{argmax}_{x\in X} f(x)$ is subcomplete.

Now suppose that X is a compact sublattice of R^n and f(x) is upper semicontinuous on X, so $\operatorname{argmax}_{x\in X} f(x)$ is nonempty and compact. By Theorem 2.7.1, $\operatorname{argmax}_{x\in X} f(x)$ is a sublattice of X and so $\operatorname{argmax}_{x\in X} f(x)$ is also a sublattice of R^n. By Theorem 2.3.1, $\operatorname{argmax}_{x\in X} f(x)$ is subcomplete. □

Theorem 2.7.2, from Milgrom and Shannon [1994], extends Theorem 2.7.1 to quasisupermodular functions.

Theorem 2.7.2. *If $f(x)$ is a quasisupermodular function from a lattice X into a chain, then* $\operatorname{argmax}_{x\in X} f(x)$ *is a sublattice of X.*

Proof. Pick any x' and x'' in $\operatorname{argmax}_{x\in X} f(x)$. Because x' is in $\operatorname{argmax}_{x\in X} f(x)$, either $f(x' \wedge x'') \prec f(x')$ or $f(x' \wedge x'') = f(x')$. If $f(x' \wedge x'') \prec f(x')$, then quasisupermodularity implies that $f(x'') \prec f(x' \vee x'')$ which contradicts x'' being in $\operatorname{argmax}_{x\in X} f(x)$. Thus, $f(x') = f(x' \wedge x'')$ and $x' \wedge x''$ is in $\operatorname{argmax}_{x\in X} f(x)$. By quasisupermodularity and $f(x') = f(x' \wedge x'')$, $f(x'') \preceq f(x' \vee x'')$ and so $x' \vee x''$ is in $\operatorname{argmax}_{x\in X} f(x)$. □

Theorem 2.7.3 and Theorem 2.7.4 state converses of Theorem 2.7.1, characterizing supermodular functions in terms of each set of maximizers of certain perturbations of a function being a sublattice. There is a tradeoff in the generality of the hypotheses for these two results. The conditions of Theorem 2.7.4 weaken those of Theorem 2.7.3 by only considering maximization over the entire sublattice domain of the function rather than over each sublattice of that sublattice domain. However, the conditions of Theorem 2.7.3 are weaker than those of Theorem 2.7.4, since the former is stated in terms of all linear perturbations of the function while the latter is in terms of all separable perturbations. See also Theorem 2.7.9.

Theorem 2.7.3. *If $f(x)$ is a real-valued function on a sublattice X of R^n and if* $\operatorname{argmax}_{x\in X'}(f(x) + t \cdot x)$ *is a sublattice of R^n for each sublattice X' of X and each t in R^n, then $f(x)$ is supermodular on X. Moreover, if $f(x)$ is a real-valued function on a sublattice X of R^n and $f(x)$ is not supermodular on X, then there* *exist unordered x' and x'' in X and t in R^n such that* $\text{argmax}_{x \in X'}(f(x) + t \cdot x) = \{x', x''\}$ *where* $X' = \{x', x'', x' \vee x'', x' \wedge x''\}$.

Proof. It suffices to prove the second statement of this result, as that implies the first statement.

Suppose that X is a sublattice of R^n and $f(x)$ is a real-valued function on X. Also, suppose that $f(x)$ is not supermodular on X, so there exist unordered x' and x'' in X with

$$f(x' \vee x'') + f(x' \wedge x'') < f(x') + f(x'')$$

and therefore

$$\begin{aligned} &f(x' \vee x'') + t \cdot (x' \vee x'') + f(x' \wedge x'') + t \cdot (x' \wedge x'') \\ &\quad < f(x') + t \cdot x' + f(x'') + t \cdot x'' \end{aligned} \tag{2.7.1}$$

for each t in R^n. Because x' and x'' are unordered, there exist integers i' and i'' with $1 \leq i' \leq n$, $1 \leq i'' \leq n$, $x'_{i'} < x''_{i'}$, and $x''_{i''} < x'_{i''}$. Define $\epsilon' = (f(x'') - f(x') + f(x' \vee x'') - f(x' \wedge x''))/2(x'_{i'} - x''_{i'})$, $\epsilon'' = (f(x'') - f(x') - f(x' \vee x'') + f(x' \wedge x''))/2(x'_{i''} - x''_{i''})$, and $t' = \epsilon' u^{i'} + \epsilon'' u^{i''}$. Then

$$\begin{aligned} &(f(x') + t' \cdot x') - (f(x'') + t' \cdot x'') \\ &\quad = f(x') - f(x'') + \epsilon'(x'_{i'} - x''_{i'}) + \epsilon''(x'_{i''} - x''_{i''}) = 0 \end{aligned} \tag{2.7.2}$$

and

$$\begin{aligned} &(f(x' \vee x'') + t' \cdot (x' \vee x'')) - (f(x' \wedge x'') + t' \cdot (x' \wedge x'')) \\ &\quad = f(x' \vee x'') - f(x' \wedge x'') + \epsilon'(x''_{i'} - x'_{i'}) + \epsilon''(x'_{i''} - x''_{i''}) = 0. \end{aligned} \tag{2.7.3}$$

Let $X' = \{x', x'', x' \vee x'', x' \wedge x''\}$. Then (2.7.1), (2.7.2), and (2.7.3) imply that $\text{argmax}_{x \in X'}(f(x) + t' \cdot x) = \{x', x''\}$. □

Theorem 2.7.4. *If $X_1, \ldots, X_n$ are chains, X is a sublattice of $\times_{i=1}^n X_i$, $f(x)$ is a real-valued function on X, there exists a real number γ such that $f(x'') - f(x') < \gamma/2$ for all x' and x'' in X, and* $\text{argmax}_{x \in X}(f(x) + w(x))$ *is a sublattice of $\times_{i=1}^n X_i$ for each separable function $w(x)$ on X, then $f(x)$ is supermodular on X.*

Proof. Suppose that $f(x)$ is not supermodular on X. By Theorem 2.6.3, there exist unordered x' and x'' in X such that

$$f(x' \vee x'') + f(x' \wedge x'') < f(x') + f(x'')$$

and

$$X \cap (\times_{i=1}^n \{x'_i, x''_i\}) = \{x', x'', x' \vee x'', x' \wedge x''\}.$$

Because x' and x'' are unordered, there exist i' and i'' with $x'_{i'} \prec x''_{i'}$ and $x''_{i''} \prec x'_{i''}$. Define the separable function $w(x) = \sum_{i=1}^{n} w_i(x_i)$ on X by

$$w_i(x_i) = \begin{cases} -\gamma & \text{if } x_i \in X_i \setminus \{x'_i, x''_i\} \\ 0 & \text{if } x_i = x'_i \\ (f(x') - f(x'') & \\ \quad - f(x' \vee x'') + f(x' \wedge x''))/2 & \text{if } i = i' \text{ and } x_i = x''_i \\ (f(x') - f(x'') & \\ \quad + f(x' \vee x'') - f(x' \wedge x''))/2 & \text{if } i = i'' \text{ and } x_i = x''_i \\ 0 & \text{if } i \neq i', i \neq i'', \text{ and } x_i = x''_i. \end{cases}$$

If x is in $X \setminus (\times_{i=1}^{n}\{x'_i, x''_i\})$, then

$$\begin{aligned} f(x) + w(x) &\leq f(x) - \gamma + \max\{w_{i'}(x''_{i'}), 0\} + \max\{w_{i''}(x''_{i''}), 0\} \\ &< f(x') = f(x') + w(x'). \end{aligned}$$

Thus,

$$\text{argmax}_{x \in X}(f(x) + w(x)) \subseteq X \cap (\times_{i=1}^{n}\{x'_i, x''_i\}) = \{x', x'', x' \vee x'', x' \wedge x''\}.$$

Furthermore,

$$f(x') + w(x') = f(x') = f(x'') + w(x''),$$

$$f(x' \vee x'') + w(x' \vee x'') = f(x' \wedge x'') + w(x' \wedge x''),$$

and

$$\begin{aligned} f(x' \vee x'') + w(x' \vee x'') &= (f(x') - f(x'') + f(x' \vee x'') + f(x' \wedge x''))/2 \\ &< f(x') = f(x') + w(x'). \end{aligned}$$

Consequently,

$$\text{argmax}_{x \in X}(f(x) + w(x)) = \{x', x''\},$$

which is not a sublattice. □

Theorem 2.7.5, from Topkis [1978], shows that the set of maxima of a strictly supermodular function is a chain. This result corresponds to the property that the set of maxima of a strictly concave function on a convex set consists of at most one point (Rockafellar [1970]).

Theorem 2.7.5. *If $f(x)$ is strictly supermodular on a lattice X, then* $\text{argmax}_{x \in X} f(x)$ *is a chain.*

Proof. Pick any x' and x'' in $\operatorname{argmax}_{x \in X} f(x)$. Suppose x' and x'' are unordered. Then because $f(x)$ is strictly supermodular on the lattice X and x' and x'' are in $\operatorname{argmax}_{x \in X} f(x)$,

$$0 \leq f(x'') - f(x' \wedge x'') < f(x' \vee x'') - f(x') \leq 0,$$

which is a contradiction. Thus $\operatorname{argmax}_{x \in X} f(x)$ has no unordered elements, and so it is a chain. □

2.7.2 Preservation of Supermodularity

Theorem 2.7.6, from Topkis [1978], states that supermodularity is preserved under the maximization operation. Likewise, concavity is preserved under the maximization operation (Dantzig [1955]). When viewed in the context of the complementary product interpretation of Section 2.6, Theorem 2.7.6 simply says that if one optimizes a system of complementary products with respect to any subset of the products then the remaining products would still be complementary. This interpretation is formalized in Example 2.7.1 and Corollary 2.7.2. One implication of Theorem 2.7.6 is a validation of the use of simplified models for a system of complementary products in which optimal decisions are made. Then, qualitative properties of a model incorporating only a strict subset of the products and assuming complementarity for those products would remain valid for that subset of products in a model including all the complementary products.

Theorem 2.7.6. *If X and T are lattices, S is a sublattice of $X \times T$, $f(x, t)$ is supermodular in (x, t) on S, S_t is the section of S at t in T, and $g(t) = \sup_{x \in S_t} f(x, t)$ is finite on the projection $\Pi_T S$ of S on T, then $g(t)$ is supermodular on $\Pi_T S$.*

Proof. Pick any t' and t'' in $\Pi_T S$, x' in $S_{t'}$, and x'' in $S_{t''}$. Because S is a sublattice of $X \times T$, $(x' \vee x'', t' \vee t'') = (x', t') \vee (x'', t'')$ is in S and $(x' \wedge x'', t' \wedge t'') = (x', t') \wedge (x'', t'')$ is in S. Thus, by the supermodularity of $f(x, t)$ and the definition of $g(t)$,

$$\begin{aligned} f(x', t') + f(x'', t'') &\leq f(x' \vee x'', t' \vee t'') + f(x' \wedge x'', t' \wedge t'') \\ &\leq g(t' \vee t'') + g(t' \wedge t''). \end{aligned} \tag{2.7.4}$$

Now taking the supremum of the left-hand side of (2.7.4) over x' in $S_{t'}$ and x'' in $S_{t''}$ yields the desired result. □

Theorem 2.7.7 is similar to Theorem 2.7.6, but considers the preservation of strict supermodularity.

Theorem 2.7.7. *If X and T are lattices, S is a sublattice of $X \times T$, $f(x,t)$ is strictly supermodular in (x,t) on S, S_t is the section of S at t in T, the maximum of $f(x,t)$ over x in S_t is attained for each t in the projection $\Pi_T S$ of S on T, and $g(t) = \max_{x \in S_t} f(x,t)$ for t in $\Pi_T S$, then $g(t)$ is strictly supermodular on $\Pi_T S$.*

Proof. Pick any unordered t' and t'' in $\Pi_T S$, x' in $S_{t'}$ with $g(t') = f(x', t')$, and x'' in $S_{t''}$ with $g(t'') = f(x'', t'')$. Because S is a sublattice of $X \times T$, $(x' \vee x'', t' \vee t'') = (x', t') \vee (x'', t'')$ is in S and $(x' \wedge x'', t' \wedge t'') = (x', t') \wedge (x'', t'')$ is in S. Because t' and t'' are unordered, (x', t') and (x'', t'') are unordered. Thus, by the strict supermodularity of $f(x,t)$ and the definition of $g(t)$,

$$\begin{aligned} g(t') + g(t'') &= f(x', t') + f(x'', t'') \\ &< f(x' \vee x'', t' \vee t'') + f(x' \wedge x'', t' \wedge t'') \\ &\le g(t' \vee t'') + g(t' \wedge t''). \quad \square \end{aligned}$$

Example 2.7.1, Example 2.7.2, and Example 2.7.3 below, as summarized in the ensuing corollaries, apply Theorem 2.7.6 to Example 2.1.1, Example 2.1.2, and Example 2.1.3, respectively. Example 2.7.2 and Corollary 2.7.3 are from Topkis [1978].

Example 2.7.1. Consider the model of Example 2.1.1, where $f(x)$ is the utility function of an n-vector x of consumption levels, I is any nonempty strict subset of the products $N = \{1, \ldots, n\}$, the consumption levels $x_{N \setminus I}$ are taken as given, and the consumer is to choose x_I to maximize the utility $f(x_I, x_{N \setminus I})$ over all x_I such that $x = (x_I, x_{N \setminus I})$ is in the constraint set X contained in R^n. Assume that X is a sublattice of R^n, $f(x)$ is supermodular on X, and $f(x)$ is bounded above on X. By Theorem 2.6.1, the n products are complements. Given I and given $x_{N \setminus I}$ in the projection of X on the coordinates $N \setminus I$, let $g(x_{N \setminus I}) = \sup_{\{x_I : (x_I, x_{N \setminus I}) \in X\}} f(x_I, x_{N \setminus I})$. By Theorem 2.7.6, $g(x_{N \setminus I})$ is supermodular in $x_{N \setminus I}$ as summarized in Corollary 2.7.2. Then by Theorem 2.6.1, the $n - |I|$ products $N \setminus I$ remain complements after optimizing over the values of the $|I|$ products I.

Corollary 2.7.2. *If X is a sublattice of R^n, $f(x)$ is supermodular and bounded above on X, I is a subset of $N = \{1, \ldots, n\}$ with $1 \le |I| \le n - 1$, and $g(x_{N \setminus I}) = \sup_{\{x_I : (x_I, x_{N \setminus I}) \in X\}} f(x_I, x_{N \setminus I})$, then $g(x_{N \setminus I})$ is supermodular in $x_{N \setminus I}$ on the projection of X on the coordinates $N \setminus I$.*

Example 2.7.2. Consider the model of Example 2.1.2, where f(x) is the real-valued utility function of an n-vector x of consumption levels, p is the n-vector of prices for acquiring (or producing) the n products, and a consumer is to choose x to maximize the net value $f(x) - p \cdot x$ over a constraint set X contained in R^n. Assume that X is a sublattice of R^n and $f(x)$ is supermodular on X. By Theorem 2.6.1, the n products are complements. Make the substitution $t = -p$. Then $f(x, t) = f(x) + t \cdot x$ is supermodular in (x, t) on $X \times R^n$ by Example 2.6.3. Assume that $f(x, t)$ is bounded above for each t in R^n, and let $g(t) = \sup_{x \in X} f(x, t)$ for t in R^n. By Theorem 2.7.6, $g(t)$ is supermodular in t. Therefore, as is summarized in Corollary 2.7.3, the function $h(p) = g(-p) = \sup_{x \in X}(f(x) - p \cdot x)$ is supermodular in p. Furthermore, $h(p)$ is convex in p because it is the supremum of a collection of affine functions (Rockafellar [1970]). Therefore, using Theorem 2.6.1, the increase in the optimal net value resulting from a decrease in some price p_i by a particular amount (that is, $h(p - \epsilon u^i) - h(p)$ where $0 < \epsilon \leq p_i$) is a decreasing function of the initial value of (each component of) p. This is a natural manifestation of complementarity, as the affordability of other products tends to reinforce benefits derived from improvements in the affordability of any particular product.

Corollary 2.7.3. *If X is a sublattice of R^n, $f(x)$ is supermodular on X, and $h(p) = \sup_{x \in X}(f(x) - p \cdot x)$ is finite for each p in R^n, then $h(p)$ is supermodular in p.*

Example 2.7.3. Consider the model of Example 2.1.3, where f(x) is the real-valued utility function of an n-vector x of consumption levels for the set of products $N = \{1, \ldots, n\}$ potentially available in the market, I is the subset of the products N actually available in the market, the consumption level of each product in $N \setminus I$ must be 0, and a consumer is to choose x_I to maximize the utility $f(x_I, x_{N \setminus I})$ over all x_I such that $x_{N \setminus I} = 0$ and $x = (x_I, x_{N \setminus I})$ is in a constraint set X contained in R^n. Assume that X is a sublattice of R^n, the n-vector 0 is in X, $x \geq 0$ for each x in X, $f(x)$ is supermodular on X, and $f(x)$ is bounded above on X. By Theorem 2.6.1, the n products are complements. Given a subset I of N, let $g(I) = \sup_{\{x: x=(x_I, x_{N \setminus I}) \in X, x_{N \setminus I}=0\}} f(x)$. Note that $\{(x, I) : x \in X, x_{N \setminus I} = 0, I \in \mathcal{P}(N)\}$ is a sublattice of $X \times \mathcal{P}(N)$ with the sublattice X of R^n having the usual ordering relation $\leq$ and the lattice $\mathcal{P}(N)$ having the ordering relation $\subseteq$. By Theorem 2.7.6, $g(I)$ is supermodular in I as summarized in Corollary 2.7.4. Then by Theorem 2.6.1, the n potentially available products N are complements in the sense that with optimal consumption decisions the additional utility from having one more product available in the market is an increasing function of the set of products already available in the market.

Corollary 2.7.4. *If X is a sublattice of R^n, the n-vector 0 is in X, $x \geq 0$ for each x in X, and $f(x)$ is supermodular and bounded above on X, then $g(I) = \sup_{\{x : x=(x_I, x_{N\setminus I}) \in X, x_{N\setminus I}=0\}} f(x)$ is supermodular in I on the power set $\mathcal{P}(N)$.*

If $f(x)$ is a function on a subset X of a set Z, then a function $h(x)$ on Z is an **extension** of $f(x)$ from X to Z if $h(x) = f(x)$ for each x in X. Theorem 2.7.8, from Topkis [1978], gives conditions under which a supermodular function on a sublattice X of R^n has an extension to a supermodular function on R^n. A corresponding extension result holds for continuously differentiable concave functions on R^n. Topkis [1978] gives further conditions under which a supermodular function has a supermodular extension.

Theorem 2.7.8. *If X is a nonempty sublattice of R^n, $f(x)$ is supermodular on X, and either X is finite or X is compact and $f(x)$ is continuously differentiable on a compact convex set containing X, then there exists a supermodular extension of $f(x)$ to R^n.*

Proof. By the above hypotheses and using Taylor's Theorem for the case where X is not finite, there exists $\gamma > 0$ such that

$$f(x) - \gamma \textstyle\sum_{i=1}^{n} |x_i - z_i| \leq f(z) \tag{2.7.5}$$

for each x and z in X. Define $g(x, z) = f(x) - \gamma \sum_{i=1}^{n} |x_i - z_i|$ for (x, z) in $X \times R^n$. Define $h(z) = \max_{x \in X} g(x, z)$ for z in R^n. Note that $h(z)$ exists because either X is finite or X is compact and $g(x, z)$ is continuous in x on X for each z in R^n. Furthermore, $h(z)$ is supermodular on R^n by Theorem 2.7.6 because, using part (g) of Example 2.6.2 and part (a) and part (b) of Lemma 2.6.1, $g(x, z)$ is supermodular on $X \times R^n$. For x and z in X, $g(x, z) \leq f(z) = g(z, z)$ by (2.7.5) and so $h(z) = f(z)$. Thus $h(x)$ is a supermodular extension of $f(x)$ from X to R^n. □

Example 2.7.4 applies Theorem 2.7.8 to establish an extension of a supermodular function on the edges of an acyclic network to a supermodular function on the edges of the complete network. (See Subsection 3.5.1.)

Example 2.7.4. Consider an acyclic network with the set of nodes $\{1, \ldots, n\}$ and the set of directed edges $T = \{(i', i'') : 1 \leq i' < i'' \leq n, i' \text{ and } i'' \text{ integer}\}$. The set T is a sublattice of R^2 as in part (d) of Example 2.2.7. By Theorem 2.7.8, there exists an extension of any supermodular function on T to a supermodular function on the edges $\{(i', i'') : 1 \leq i' \leq n, 1 \leq i'' \leq n, i' \text{ and } i'' \text{ integer}\}$ of the full network with a directed edge joining any ordered pair of nodes.

Theorem 2.7.9 gives a converse of Theorem 2.7.1, showing that, with a few regularity conditions, any sublattice of R^n can be represented as the set of unconstrained maxima of a supermodular function on R^n. See Theorem 3.7.7 for a version of this result for finite lattices.

Theorem 2.7.9. *If X is a nonempty compact sublattice of R^n, then there exists a supermodular function $f(x)$ on R^n such that $X = \text{argmax}_{x \in R^n} f(x)$.*

Proof. For x in R^n and z in R^n, define $g(x, z) = -\sum_{i=1}^{n} |x_i - z_i|$. Because $g(x, z)$ is continuous and X is nonempty and compact, the maximum of $g(x, z)$ over z in X is attained for each x in R^n. Let $f(x) = \max_{z \in X} g(x, z)$ for x in R^n. By Corollary 2.6.1, $g(x, z)$ is supermodular in (x, z) on R^{2n} as in part (g) of Example 2.6.2. Therefore, $f(x)$ is supermodular on R^n by Theorem 2.7.6. If x is in X, then $f(x) = g(x, x) = 0$. If x is in R^n but not in X, then $f(x) = g(x, z) < 0$ for some z in X. Therefore, $\text{argmax}_{x \in R^n} f(x) = X$. □

2.8 Increasing Optimal Solutions

This section explores necessary and sufficient conditions for a parameterized collection of optimization problems as in (2.1.1) to have increasing optimal solutions. Increasing optimal solutions correspond to the notion of complementarity between a decision variable x and a parameter t, where a higher level of t results in higher optimal levels for x. (This notion of complementarity is not entirely equivalent to that defined in Subsection 2.6.1. Connections between these and other notions of complementarity are elucidated in the present section and in Section 2.9.) Subsection 2.8.1 gives sufficient conditions. Subsection 2.8.2 gives necessary conditions.

2.8.1 Sufficient Conditions

Consider the collection of parameterized optimization problems (2.1.1), maximize $f(x, t)$ subject to x being in a subset S_t of X, where both the constraint set S_t and the objective function $f(x, t)$ depend on the parameter t for t in a set T. Here, X and T are partially ordered sets and $\text{argmax}_{x \in S_t} f(x, t)$ is the set of optimal solutions for (2.1.1) given t in T. This subsection gives sufficient conditions for increasing optimal solutions (that is, for $\text{argmax}_{x \in S_t} f(x, t)$ to be increasing in t on T where $\mathcal{P}(X) \setminus \{\emptyset\}$ has the induced set ordering $\sqsubseteq$) and for there to exist increasing optimal selections (that is, selections of an element from each nonempty $\text{argmax}_{x \in S_t} f(x, t)$ so that the element is increasing in t on $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t) \text{ is nonempty}\}$). Sufficient conditions may typically be stated only for increasing optimal solutions (as with Lemma 2.8.1, Theorem 2.8.1, and Theorem 2.8.2) because Theorem 2.4.3 and Theorem 2.4.4 provide standard regularity conditions under which having increasing optimal solutions for a parameterized collection of optimization problems directly implies the existence of increasing optimal selections (as in Theorem 2.8.3).

The sufficient conditions of this subsection are composed of three groups. First, there are sufficient conditions for increasing optimal solutions (Lemma 2.8.1, Theorem 2.8.1, Theorem 2.8.2) and increasing optimal selections (Theorem 2.8.3, Theorem 2.8.4, Theorem 2.8.5) based on supermodularity and increasing differences. Next, there are applications of these sufficient conditions (Example 2.8.1, Example 2.8.2, Corollary 2.8.1, Example 2.8.3, Corollary 2.8.2, Example 2.8.4, Corollary 2.8.3) to the LeChatelier principle and to examples involving systems of complementary products with parameters being, respectively, the consumption levels of some particular subset of the products, the vector of prices of the products, and the subset of the products available to consumers in the market. Finally, there are sufficient conditions for increasing optimal solutions (Theorem 2.8.6) and increasing optimal selections (Theorem 2.8.7) based on the ordinal conditions of quasisupermodularity and the single crossing property.

Lemma 2.8.1, from Topkis [1978], offers a general statement of sufficient conditions for increasing optimal solutions, but these conditions are not in the most convenient form. See comments in the paragraph preceding Theorem 2.8.2. (In Lemma 2.8.1, the assumption that S_t is increasing in t implies that each S_t is a nonempty sublattice as discussed in Section 2.4. The implicit assumption that each S_t is a sublattice is only used because the conclusion that $\text{argmax}_{x \in S_t} f(x, t)$ is increasing in t requires that $\text{argmax}_{x \in S_t} f(x, t) \sqsubseteq \text{argmax}_{x \in S_t} f(x, t)$ for each t and hence $\text{argmax}_{x \in S_t} f(x, t)$ must be a sublattice for each t.)

Lemma 2.8.1. *If X is a lattice, T is a partially ordered set, S_t is a subset of X for each t in T, S_t is increasing in t on T, and*

$$f(x', t') + f(x'', t'') \le f(x' \wedge x'', t') + f(x' \vee x'', t'') \tag{2.8.1}$$

for all t' and t'' in T with $t' \preceq t''$, x' in $S_{t'}$, and x'' in $S_{t''}$, then $\text{argmax}_{x \in S_t} f(x, t)$ *is increasing in t on $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t)$ is nonempty$\}$.*

Proof. By (2.8.1) and Theorem 2.7.1, $\text{argmax}_{x \in S_t} f(x, t)$ is a sublattice of X for each t in T. Pick t' and t'' in $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t) \text{ is nonempty}\}$ with $t' \preceq t''$. Then pick x' in $\text{argmax}_{x \in S_{t'}} f(x, t')$ and x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. Because $S_{t'} \sqsubseteq S_{t''}$, $x' \wedge x''$ is in $S_{t'}$ and $x' \vee x''$ is in $S_{t''}$. Then by (2.8.1) and the optimality of x' and x'' for t' and t'', respectively,

$$0 \le f(x', t') - f(x' \wedge x'', t') \le f(x' \vee x'', t'') - f(x'', t'') \le 0. \tag{2.8.2}$$

Thus equality holds throughout in (2.8.2), and so $x' \wedge x''$ is in $\text{argmax}_{x \in S_{t'}} f(x, t')$ and $x' \vee x''$ is in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. □

Theorem 2.8.1, from Topkis [1978], is a direct consequence of Lemma 2.8.1 and is more convenient to use. This result gives sufficient conditions, in terms of supermodularity and increasing differences, for a parameterized collection of optimization problems to have increasing optimal solutions.

Theorem 2.8.1. *If X is a lattice, T is a partially ordered set, S_t is a subset of X for each t in T, S_t is increasing in t on T, $f(x, t)$ is supermodular in x on X for each t in T, and $f(x, t)$ has increasing differences in (x, t) on $X \times T$, then* $\text{argmax}_{x \in S_t} f(x, t)$ *is increasing in t on $\{t : t \in T,$* $\text{argmax}_{x \in S_t} f(x, t)$ *is nonempty$\}$.*

Proof. By Lemma 2.8.1, it suffices to show that (2.8.1) holds. Pick t' and t'' in T with $t' \preceq t''$, x' in $S_{t'}$, and x'' in $S_{t''}$. Then because $f(x, t)$ is supermodular in x and has increasing differences in (x, t),

$$\begin{aligned} f(x', t') - f(x' \wedge x'', t') &\le f(x' \vee x'', t') - f(x'', t') \\ &\le f(x' \vee x'', t'') - f(x'', t'') \end{aligned}$$

verifying that (2.8.1) holds. □

When X is a product set in R^n and T is a product set in R^m, the conditions of Theorem 2.8.1 hold by Theorem 2.6.1 and Corollary 2.6.1 if and only if each pair of components of the decision variable x are complements and each component of the decision variable x is complementary with each component of the parameter t. The components of the parameter t need not be complements. (The conditions on the components of the decision variable are stronger than the conditions on the components of the parameter. Thus, if the conditions of Theorem 2.8.1 hold for a particular collection of parameterized optimization problems and if the structure of these optimization problems is altered transferring some component of the decision variable into a new component of the parameter, then the conditions of Theorem 2.8.1 would hold for the newly altered collection of parameterized optimization problems.)

By part (a) of Theorem 2.4.5, the assumption in Lemma 2.8.1 and in Theorem 2.8.1 that S_t is increasing in t on T holds if T as well as X is a lattice, S is a sublattice of $X \times T$, $S_t = \{x : (x, t) \in S\}$ is the section of S at t in T, and T is the projection of S onto T. (When T is a chain, part (b) of Theorem 2.4.5 further implies that the latter conditions are equivalent to S_t increasing in t.) Furthermore, assumption (2.8.1) in Lemma 2.8.1 holds if $\{(x, t) : x \in S_t, t \in T\}$ is a sublattice of $X \times T$ and $f(x, t)$ is supermodular in (x, t) on $\{(x, t) : x \in S_t, t \in T\}$. These observations immediately lead to Theorem 2.8.2, from Topkis [1978], as a consequence of Lemma 2.8.1. Theorem 2.8.2 and Lemma 2.8.1 are equivalent if T is a chain. Theorem 2.8.2 provides succinct and convenient sufficient conditions for increasing optimal solutions. However, the hypotheses of Theorem 2.8.2 include the superfluous conditions that the set of parameters $\{t : x \in S_t, t \in T\}$ is a sublattice of T for each x in X and that $f(x, t)$ is supermodular in the parameter t on $\{t : x \in S_t, t \in T\}$ for each x in X. (See also Example 2.4.3.) On the other hand, the sufficient conditions of Theorem 2.8.1, unlike those of Lemma 2.8.1 and Theorem 2.8.2, may require that $f(x, t)$ have certain properties of a set larger than $S = \{(x, t) : t \in T, x \in S_t\}$.

Theorem 2.8.2. *If X and T are lattices, S is a sublattice of $X \times T$, S_t is the section of S at t in T, and $f(x, t)$ is supermodular in (x, t) on S, then* $\text{argmax}_{x \in S_t} f(x, t)$ *is increasing in t on* $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t)$ *is nonempty*$\}$.

Suppose that X is a subset of R^n, T is a subset of R^m, and $f(x, t)$ is twice differentiable on a convex product set S that contains $X \times T$ in R^{n+m}. Then the conditions on $f(x, t)$ in Theorem 2.8.1 with S_t being the section of S at t in T hold by Corollary 2.6.1 if, for each (x, t) in S, $\partial^2 f(x, t)/\partial x_{i'} \partial x_{i''} \geq 0$ for all $i' \neq i''$ and $\partial^2 f(x, t)/\partial x_i \partial t_j \geq 0$ for all i and j. The conditions on $f(x, t)$ in Theorem 2.8.2 would hold by Corollary 2.6.1 if, in addition to the conditions for Theorem 2.8.1, $\partial^2 f(x, t)/\partial t_{j'} \partial t_{j''} \geq 0$ for all $j' \neq j''$ and each (x, t) in S.

Theorem 2.8.3 uses the sufficient conditions given above for increasing optimal solutions to state corresponding sufficient conditions for the existence of increasing optimal selections. Part (a), from Topkis [1978], is a consequence of Corollary 2.7.1, Theorem 2.4.3, Theorem 2.8.1, and Theorem 2.8.2. Part (b) is a consequence of Theorem 2.4.4, Theorem 2.8.1, and Theorem 2.8.2. Part (a) imposes stronger regularity conditions than part (b) on the objective function and on the constraint sets, while part (b) has the added requirement that either the set of decision variables or the parameter set be finite.

Theorem 2.8.3. *Suppose that the hypotheses of either Theorem* 2.8.1 *or Theorem* 2.8.2 *hold and each S_t is nonempty.*
(a) *If either each S_t is finite or each S_t is a compact subset of R^n and $f(x, t)$ is upper semicontinuous in x on S_t for each t in T, then each* $\text{argmax}_{x \in S_t} f(x, t)$ (*being either finite or a compact subset of R^n*) *is a nonempty sublattice of X with a greatest element and a least element and the greatest* (*least*) *element of* $\text{argmax}_{x \in S_t} f(x, t)$ *is increasing in t on T.*
(b) *If either X or T is finite and x'_t is in* $\text{argmax}_{x \in S_t} f(x, t)$ *for each t in T, then* $\sup\{x'_\tau : \tau \in T, \tau \preceq t\}$ $(\inf\{x'_\tau : \tau \in T, t \preceq \tau\})$ *is an increasing optimal selection from* $\text{argmax}_{x \in S_t} f(x, t)$.

Example 2.8.1 is a version of the LeChatelier principle from Milgrom and Roberts [1996], who give a more general statement based on the ordinal conditions of Theorem 2.8.6.

Example 2.8.1. A firm has two decision variables, x chosen from a nonempty compact sublattice X of R^n and z chosen from a nonempty compact sublattice Z of R^m. The firm has a real-valued profit function $f(x, z, t)$, which depends on the decision variables x and z as well as on a parameter t from a chain T. Assume that $f(x, z, t)$ is upper semicontinuous in (x, z) on $X \times Z$ for each t in T and is supermodular in (x, z, t) on $X \times Z \times T$. The firm optimizes the decision variables x and z over X and Z, respectively, given any parameter t in T. However, when the parameter t changes, the firm can change x in the short run but can only change z in the long run. The firm always selects the greatest (least) optimal short run and long run decisions (which exist by Corollary 2.7.1). Suppose that the firm has adjusted (in the long run) to a parameter t' with decision variables (x', z'), where (x', z') is the greatest element of $\text{argmax}_{x\in X, z\in Z} f(x, z, t')$ and so x' is the greatest element of $\text{argmax}_{x\in X} f(x, z', t')$. Consider a change in the parameter from t' to t'', where $t' \prec t''$. The firm's optimal short run decision is (x'', z'), where x'' is the greatest element of $\text{argmax}_{x\in X} f(x, z', t'')$. The firm's optimal long run decision is (x''', z''), where (x''', z'') is the greatest element of $\text{argmax}_{x\in X, z\in Z} f(x, z, t'')$ and so x''' is the greatest element of $\text{argmax}_{x\in X} f(x, z'', t'')$. Because (x', z') is the greatest element of $\text{argmax}_{x\in X, z\in Z} f(x, z, t')$, (x''', z'') is the greatest element of $\text{argmax}_{x\in X, z\in Z} f(x, z, t'')$, and $t' \prec t''$, part (a) of Theorem 2.8.3 implies that $(x', z') \leq (x''', z'')$. Because x' is the greatest element of $\text{argmax}_{x\in X} f(x, z', t')$, x'' is the greatest element of $\text{argmax}_{x\in X} f(x, z', t'')$, x''' is the greatest element of $\text{argmax}_{x\in X} f(x, z'', t'')$, $z' \leq z''$, and $t' \prec t''$, part (a) of Theorem 2.8.3 implies that $x' \leq x'' \leq x'''$. Thus, as the parameter t increases, the decision variable x increases in the short run and increases again in the long run.

Theorem 2.8.4, from Topkis [1978], shows that if the hypothesis of increasing differences in Theorem 2.8.1 is strengthened to strictly increasing differences, then there is a stronger ordering relation than $\sqsubseteq$ on the sets $\text{argmax}_{x\in S_t} f(x, t)$. (With this stronger ordering relation, two subsets X' and X'' of some partially ordered set would be ordered with X'' greater than X' if $x' \preceq x''$ for all x' in X' and x'' in X''. This binary relation, implicit in the second sentence of Theorem 2.8.4, is antisymmetric and transitive on $\mathcal{P}(X) \setminus \{\emptyset\}$ for any partially ordered set X; however, it is only reflexive on those subsets of X having no more that one element.) Here, constructing an increasing optimal selection need not involve selecting particular elements from each $\text{argmax}_{x\in S_t} f(x, t)$

because any selection from $\text{argmax}_{x \in S_t} f(x, t)$ is increasing. Contrast this with part (a) of Theorem 2.8.3, where the greatest element or the least element from each of these sets is the increasing optimal selection, and with part (b) of Theorem 2.8.3, where a certain transformation must be performed on some optimal selection to generate an increasing optimal selection.

Theorem 2.8.4. *Suppose that X is a lattice, T is a partially ordered set, S_t is a subset of X for each t in T, S_t is increasing in t on T, $f(x, t)$ is supermodular in x on X for each t in T, and $f(x, t)$ has strictly increasing differences in (x, t) on $X \times T$. If t' and t'' are in T, $t' \prec t''$, x' is in* $\text{argmax}_{x \in S_{t'}} f(x, t')$, *and x'' is in* $\text{argmax}_{x \in S_{t''}} f(x, t'')$, *then $x' \preceq x''$. Hence, if one picks any x_t in* $\text{argmax}_{x \in S_t} f(x, t)$ *for each t in T with* $\text{argmax}_{x \in S_t} f(x, t)$ *nonempty, then x_t is increasing in t on* $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t)$ *is nonempty*$\}$.

Proof. Pick t' and t'' in T with $t' \prec t''$, x' in $\text{argmax}_{x \in S_{t'}} f(x, t')$, and x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. Because $S_{t'} \sqsubseteq S_{t''}$, $x' \wedge x''$ is in $S_{t'}$ and $x' \vee x''$ is in $S_{t''}$. Suppose it is not true that $x' \preceq x''$. Then $x'' \prec x' \vee x''$ and so using the hypotheses that $f(x, t)$ is supermodular in x and has strictly increasing differences in (x, t),

$$\begin{aligned} 0 &\le f(x', t') - f(x' \wedge x'', t') \le f(x' \vee x'', t') - f(x'', t') \\ &< f(x' \vee x'', t'') - f(x'', t'') \le 0, \end{aligned}$$

which is a contradiction. □

Theorem 2.8.5, from Amir [1996], strengthens the hypotheses of Theorem 2.8.4 to give conditions for optimal selections to be strictly increasing. Edlin and Shannon [1996] establish conditions for optimal selections to be strictly increasing where the objective function satisfies ordinal hypotheses rather than the present cardinal hypotheses.

Theorem 2.8.5. *Suppose that X is a convex sublattice of R^n, T is a partially ordered set, S_t is a subset of X for each t in T, S_t is increasing in t on T, $f(x, t)$ is a real-valued function on $X \times T$ that is differentiable and supermodular in x on X for each t in T, and $\partial f(x, t)/\partial x_i$ is strictly increasing in t for all (x, t) in $X \times T$ and $i = 1, \ldots, n$. If t' and t'' are in T, $t' \prec t''$, x' is in* $\text{argmax}_{x \in S_{t'}} f(x, t')$, *$x''$ is in* $\text{argmax}_{x \in S_{t''}} f(x, t'')$, *and i is such that there exists $\epsilon' > 0$ with $x' + \epsilon u^i$ in $S_{t'}$ ($x'' + \epsilon u^i$ in $S_{t''}$) for each real ϵ with $|\epsilon| \le \epsilon'$, then $x' \prec x''$ and $x'_i \prec x''_i$.*

Proof. By hypotheses, $f(x, t)$ has strictly increasing differences in (x, t) on $X \times T$. Let t', t'', x', x'', i, and ϵ' be as in the statement of this result. By Theorem 2.8.4, $x' \preceq x''$ and so it suffices to show that $x'_i \prec x''_i$. Suppose that $x'_i = x''_i$. Because $x' + \epsilon u^i$ is in $S_{t'}$ for each real ϵ with $0 \le \epsilon \le \epsilon'$, $S_{t'} \sqsubseteq S_{t''}$

implies that $(x' + \epsilon u^i) \vee x'' = x'' + \epsilon u^i$ is in $S_{t''}$ for each real ϵ with $0 \leq \epsilon \leq \epsilon'$. By hypotheses,

$$0 = \partial f(x', t')/\partial x_i \leq \partial f(x'', t')/\partial x_i < \partial f(x'', t'')/\partial x_i.$$

But this contradicts x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$, so $x'_i \prec x''_i$.

The proof for the case in parentheses is similar. □

Example 2.8.2, Example 2.8.3, and Example 2.8.4 present cases of Example 2.1.1, Example 2.1.2, and Example 2.1.3, respectively, where applications of the above sufficient conditions for increasing optimal solutions and increasing optimal selections yield natural manifestations of complementarity. Formal versions of these results are given in Corollary 2.8.1, Corollary 2.8.2, and Corollary 2.8.3.

Example 2.8.2 and Corollary 2.8.1 show that with a system of complementary products if the consumption levels of any subset of the products are treated as a parameter and held fixed then the optimal consumption levels of all the other products are increasing with the parametric consumption levels of the former subset of the products. This provides a natural interpretation of Theorem 2.8.1 and Theorem 2.8.2 in terms of complementarity. According to the definition of complementarity based on increasing differences, the marginal value of each product increases with the consumption level of each other product. By Theorem 2.8.1 and Theorem 2.8.2, optimal consumption levels of any subset of a system of complementary products increase with the consumption levels of the other products. In both cases (increasing differences and increasing optimal solutions), greater consumption of any subset of products is more desirable for higher consumption levels of the other products. The following shows that the former (increasing differences) implies the latter notion (increasing optimal solutions) of increased desirability.

Example 2.8.2. Consider the model of Example 2.1.1, where $f(x)$ is the real-valued utility function of an n-vector x of consumption levels, I is any nonempty strict subset of the products $N = \{1, \ldots, n\}$, the consumption levels $x_{N \setminus I}$ of the products $N \setminus I$ are taken as given, and a consumer is to choose the consumption levels x_I of the products I to maximize the utility $f(x_I, x_{N \setminus I})$ over all x_I such that $x = (x_I, x_{N \setminus I})$ is in a constraint set X contained in R^n. If X is a sublattice of R^n and $f(x)$ is supermodular on X, then Theorem 2.8.2 implies that there are increasing optimal solutions x_I as a function of $x_{N \setminus I}$. If, in addition, X is nonempty and compact and $f(x)$ is upper semicontinuous, then part (a) of Theorem 2.8.3 implies that there are increasing optimal selections x_I as a function of $x_{N \setminus I}$. This is summarized in Corollary 2.8.1. Therefore, for a system with n complementary products, the optimal consumption levels of any k products given fixed consumption levels of the other $n-k$ products are increasing as a function of the consumption levels of the latter $n-k$ products.

Corollary 2.8.1. *If X is a sublattice of R^n, $f(x)$ is supermodular on X, and I is a subset of $N = \{1, \ldots, n\}$ with $1 \leq |I| \leq n-1$, then* $\text{argmax}_{\{x_I : (x_I, x_{N\setminus I}) \in X\}} f(x_I, x_{N\setminus I})$ *is increasing in $x_{N\setminus I}$ for $x_{N\setminus I}$ such that* $\text{argmax}_{\{x_I : (x_I, x_{N\setminus I}) \in X\}} f(x_I, x_{N\setminus I})$ *is nonempty. If, in addition, X is nonempty and compact and $f(x)$ is upper semicontinuous on X, then each* $\text{argmax}_{\{x_I : (x_I, x_{N\setminus I}) \in X\}} f(x_I, x_{N\setminus I})$ *for $x_{N\setminus I}$ in the projection of X onto the coordinates $N \setminus I$ has a greatest element and a least element and the greatest (least) element of* $\text{argmax}_{\{x_I : (x_I, x_{N\setminus I}) \in X\}} f(x_I, x_{N\setminus I})$ *is increasing in $x_{N\setminus I}$ on the projection of X onto the coordinates $N \setminus I$.*

Example 2.8.3 and Corollary 2.8.2, from Topkis [1978], show that with a system of complementary products and linear acquisition costs for the products the optimal consumption levels of all products decrease with an increase in the acquisition price of any product. This monotonicity property with parametric prices is sometimes taken as a definition of complementary products. (See Theorem 2.8.8 for a converse statement.)

Example 2.8.3. Consider the model of Example 2.1.2 involving the determination of optimal consumption levels with a price parameter, where $f(x)$ is the real-valued utility function of an n-vector x of consumption levels, p is the n-vector of prices for the n products, and a consumer is to choose x to maximize the net value $f(x) - p \cdot x$ over a constraint set X contained in R^n. If X is a sublattice of R^n and $f(x)$ is supermodular on X, then Theorem 2.8.2 and Example 2.6.3 imply that there are decreasing optimal solutions for the consumption levels x as a function of the price vector p. If, in addition, X is nonempty and compact and $f(x)$ is upper semicontinuous, then part (a) of Theorem 2.8.3 and Example 2.6.3 imply that there are decreasing optimal selections for the consumption levels x as a function of the price vector p. This is summarized in Corollary 2.8.2.

Corollary 2.8.2. *If X is a sublattice of R^n and $f(x)$ is supermodular on X, then* $\text{argmax}_{x \in X}(f(x) - p \cdot x)$ *is decreasing in p for those p in R^n with* $\text{argmax}_{x \in X}(f(x) - p \cdot x)$ *nonempty. If, in addition, X is nonempty and compact and $f(x)$ is upper semicontinuous on X, then each* $\text{argmax}_{x \in X}(f(x) - p \cdot x)$ *has a greatest element and a least element and the greatest (least) element of* $\text{argmax}_{x \in X}(f(x) - p \cdot x)$ *is decreasing in p on R^n.*

Example 2.8.4 and Corollary 2.8.3 show that when products are complementary and only some subset of potentially available products is actually available in the market, the optimal consumption levels of all products increase with the set of available products. That is, when any new product is introduced into the market, the consumption of all other existing products increases.

Example 2.8.4. Consider the model of Example 2.1.3, where $f(x)$ is the real-valued utility function of an n-vector x of consumption levels, I is a subset of the products $N = \{1, \ldots, n\}$, the consumption level of each product in $N \setminus I$ must be 0, and a consumer is to choose x_I to maximize the utility $f(x)$ over all x_I such that $x = (x_I, x_{N\setminus I})$ is in a constraint set X contained in R^n and $x_{N\setminus I} = 0$. If X is a sublattice of R^n, the n-vector 0 is in X, $x \geq 0$ for each x in X, and $f(x)$ is supermodular on X, then Theorem 2.8.2 implies that there are increasing optimal solutions for the consumption levels x as a function of the set I of products available in the market. If, in addition, X is compact and $f(x)$ is upper semicontinuous, then part (a) of Theorem 2.8.3 implies that there are increasing optimal selections for the consumption levels x as a function of the set I of available products. This is summarized in Corollary 2.8.3.

Corollary 2.8.3. *If X is a sublattice of R^n, the n-vector 0 is in X, $x \geq 0$ for each x in X, $N = \{1, \ldots, n\}$, and $f(x)$ is supermodular on X, then $\text{argmax}_{\{x: x\in X, x_{N\setminus I}=0\}} f(x)$ is increasing in I for those subsets I of N with $\text{argmax}_{\{x: x\in X, x_{N\setminus I}=0\}} f(x)$ nonempty. If, in addition, X is compact and $f(x)$ is upper semicontinuous on X, then each $\text{argmax}_{\{x: x\in X, x_{N\setminus I}=0\}} f(x)$ has a greatest element and a least element and the greatest (least) element of $\text{argmax}_{\{x: x\in X, x_{N\setminus I}=0\}} f(x)$ is increasing in I on $\mathcal{P}(N)$.*

Taking a strictly increasing transformation of the objective function for optimization problems preserves sets of optimal solutions but does not preserve cardinal properties like supermodularity and increasing differences. (See Example 2.6.5.) Hence, the sufficient conditions based on supermodularity and increasing differences for increasing optimal solutions as in Theorem 2.8.1 are not necessary conditions in general. Theorem 2.8.6, from Milgrom and Shannon [1994], uses quasisupermodularity and the single crossing property to generalize the sufficient conditions of Theorem 2.8.1 for increasing optimal solutions. Just as Theorem 2.8.3 takes the conclusion of Theorem 2.8.1 from increasing optimal solutions to increasing optimal selections, one could use Theorem 2.4.3 or Theorem 2.4.4 to state a version of Theorem 2.8.6 giving sufficient conditions for increasing optimal selections.

Theorem 2.8.6. *If X is a lattice, T is a partially ordered set, S_t is a subset of X for each t in T, S_t is increasing in t on T, $f(x, t)$ is quasisupermodular in x on X for each t in T, and $f(x, t)$ satisfies the single crossing property* *in* (x, t) *on* $X \times T$, *then* $\text{argmax}_{x \in S_t} f(x, t)$ *is increasing in* t *on* $\{t : t \in T,$ $\text{argmax}_{x \in S_t} f(x, t)$ *is nonempty*$\}$.

Proof. Pick t' and t'' in T with $t' \preceq t''$, x' in $\text{argmax}_{x \in S_{t'}} f(x, t')$, and x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. Because $S_{t'} \sqsubseteq S_{t''}$, $x' \wedge x''$ is in $S_{t'}$ and $x' \vee x''$ is in $S_{t''}$. Therefore, $f(x' \wedge x'', t') \preceq f(x', t')$ and $f(x' \vee x'', t'') \preceq f(x'', t'')$. Then $f(x'', t') \preceq f(x' \vee x'', t')$ by quasisupermodularity and so $f(x'', t'') \preceq f(x' \vee x'', t'')$ by the single crossing property. Thus, $x' \vee x''$ is in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. If $f(x' \wedge x'', t') \prec f(x', t')$, then $f(x'', t') \prec f(x' \vee x'', t')$ by quasisupermodularity and so $f(x'', t'') \prec f(x' \vee x'', t'')$ by the single crossing property. But this contradicts x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. Therefore, $f(x' \wedge x'', t') = f(x', t')$ and $x' \wedge x''$ is in $\text{argmax}_{x \in S_{t'}} f(x, t')$. □

Theorem 2.8.7, from Milgrom and Shannon [1994], gives an ordinal generalization of the result of Theorem 2.8.4.

Theorem 2.8.7. *Suppose that* X *is a lattice,* T *is a partially ordered set,* S_t *is a subset of* X *for each* t *in* T, S_t *is increasing in* t *on* T, $f(x, t)$ *is quasisupermodular in* x *on* X *for each* t *in* T, *and* $f(x, t)$ *satisfies the strict single crossing property in* (x, t) *on* $X \times T$. *If* t' *and* t'' *are in* T, $t' \prec t''$, x' *is in* $\text{argmax}_{x \in S_{t'}} f(x, t')$, *and* x'' *is in* $\text{argmax}_{x \in S_{t''}} f(x, t'')$, *then* $x' \preceq x''$. *Hence, if one picks any* x_t *in* $\text{argmax}_{x \in S_t} f(x, t)$ *for each* t *in* T *with* $\text{argmax}_{x \in S_t} f(x, t)$ *nonempty, then* x_t *is increasing in* t *on* $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t)$ *is nonempty*$\}$.

Proof. Pick t' and t'' in T with $t' \prec t''$, x' in $\text{argmax}_{x \in S_{t'}} f(x, t')$, and x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. Because $S_{t'} \sqsubseteq S_{t''}$, $x' \wedge x''$ is in $S_{t'}$ and $x' \vee x''$ is in $S_{t''}$. Therefore, $f(x' \wedge x'', t') \preceq f(x', t')$ and $f(x' \vee x'', t'') \preceq f(x'', t'')$. Suppose it is not true that $x' \preceq x''$, so $x'' \prec x' \vee x''$. Then $f(x'', t') \preceq f(x' \vee x'', t')$ by quasisupermodularity, and so $f(x'', t'') \prec f(x' \vee x'', t'')$ by the strict single crossing property. But this contradicts x'' in $\text{argmax}_{x \in S_{t''}} f(x, t'')$. Therefore, $x' \preceq x''$. □

2.8.2 Necessary Conditions

The necessary conditions in this subsection consist of four types of results. Three are somewhat different forms of necessary conditions implying the cardinal properties of supermodularity and increasing differences for the objective function. The first involves increasing optimal selections with respect to a vector of prices assigned to the components of the decision variable where a corresponding price term is included with the objective function (Theorem 2.8.8). The second shows that any increasing collection of sets is identical with the sets of optimal solutions for *some* parameterized collection of maximization problems with a supermodular objective function (Theorem 2.8.9). The third involves increasing optimal selections for every collection of parameterized optimization problems where the objective function is the original objective function under consideration perturbed by an arbitrary linear or separable function (Theorem 2.8.10, Theorem 2.8.11, Theorem 2.8.12). The cardinal conditions of supermodularity and increasing differences are not necessary for increasing optimal solutions in a particular maximization problem because a strictly increasing transformation of the objective function preserves sets of optimal solutions but does not generally preserve the cardinal conditions. Necessary conditions for increasing optimal solutions in a particular parameterized collection of maximization problems involve the ordinal properties of quasisupermodularity and the single crossing property (Theorem 2.8.13, Theorem 2.8.14).

Theorem 2.8.8 states a converse of Corollary 2.8.2. For the utility function of a consumer choosing the consumption levels of n products, this result together with Corollary 2.8.2 characterizes supermodularity in terms of whether optimal consumption levels for all products are decreasing with the prices of all the products. Theorem 2.8.8 is stated in terms of the existence of an increasing optimal selection, which is weaker that having the sets of maximizers $\operatorname{argmax}_{x \in X'}(f(x) - p \cdot x)$ be decreasing in price p.

Theorem 2.8.8. *If $f(x)$ is a real-valued function on a sublattice X of R^n and if for each sublattice X' of X there exists an increasing selection with respect to t from* $\operatorname{argmax}_{x \in X'}(f(x)+t\cdot x)$ *for any two distinct t in T with* $\operatorname{argmax}_{x\in X'}(f(x)+t \cdot x)$ *nonempty, then $f(x)$ is supermodular on X.*

Proof. Suppose that X is a sublattice of R^n and $f(x)$ is a real-valued function on X. Also, suppose that $f(x)$ is not supermodular on X, so, by Theorem 2.7.3, there exist unordered x' and x'' in X and t in R^n such that $\operatorname{argmax}_{x \in X'}(f(x) + t \cdot x) = \{x', x''\}$ where $X' = \{x', x'', x' \vee x'', x' \wedge x''\}$. Because x' and x'' are unordered, there exists an integer i' with $1 \le i' \le n$ and $x'_{i'} < x''_{i'}$. Pick any $\delta > 0$ and let $t' = t - \delta u^{i'}$ and $t'' = t + \delta u^{i'}$.

Because $\operatorname{argmax}_{x \in X'}(f(x) + t \cdot x) = \{x', x''\}$, $x'_{i'} < x''_{i'}$, and $\delta > 0$, it follows that

$$\begin{aligned} 0 < \delta(x''_{i'} - x'_{i'}) &= (f(x') + t \cdot x' - \delta x'_{i'}) - (f(x'') + t \cdot x'' - \delta x''_{i'}) \\ &= (f(x') + t' \cdot x') - (f(x'') + t' \cdot x''), \end{aligned}$$

$$\begin{aligned} 0 < \delta(x''_{i'} - x'_{i'}) &< (f(x') + t \cdot x' - \delta x'_{i'}) - (f(x' \vee x'') + t \cdot (x' \vee x'') - \delta x''_{i'}) \\ &= (f(x') + t' \cdot x') - (f(x' \vee x'') + t' \cdot (x' \vee x'')), \end{aligned}$$

$$0 < (f(x') + t \cdot x' - \delta x'_{i'}) - (f(x' \wedge x'') + t \cdot (x' \wedge x'') - \delta x'_{i'})$$
$$= (f(x') + t' \cdot x') - (f(x' \wedge x'') + t' \cdot (x' \wedge x'')),$$

$$0 < \delta(x''_{i'} - x'_{i'}) < (f(x'') + t \cdot x'' + \delta x''_{i'}) - (f(x') + t \cdot x' + \delta x'_{i'})$$
$$= (f(x'') + t'' \cdot x'') - (f(x') + t'' \cdot x'),$$

$$0 < (f(x'') + t \cdot x'' + \delta x''_{i'}) - (f(x' \vee x'') + t \cdot (x' \vee x'') + \delta x''_{i'})$$
$$= (f(x'') + t'' \cdot x'') - (f(x' \vee x'') + t'' \cdot (x' \vee x'')),$$

and

$$0 < \delta(x''_{i'} - x'_{i'}) < (f(x'') + t \cdot x'' + \delta x''_{i'}) - (f(x' \wedge x'') + t \cdot (x' \wedge x'') + \delta x'_{i'})$$
$$= (f(x'') + t'' \cdot x'') - (f(x' \wedge x'') + t'' \cdot (x' \wedge x'')).$$

Therefore, $\text{argmax}_{x \in X'}(f(x) + t' \cdot x) = \{x'\}$ and $\text{argmax}_{x \in X'}(f(x) + t'' \cdot x) = \{x''\}$.

Because $t' < t''$ and x' and x'' are unordered, no increasing selection from $\text{argmax}_{x \in X'}(f(x) + t \cdot x)$ exists. □

Theorem 2.8.9 shows that, with a few regularity conditions, any increasing collection of sets can be represented as the sets of unconstrained maxima of a supermodular function.

Theorem 2.8.9. *If T is a chain, S_t is a compact subset of R^n for each t in T, and S_t is increasing in t on T, then there exists a supermodular function $f(x, t)$ on $R^n \times T$ such that* $\text{argmax}_{x \in R^n} f(x, t) = S_t$ *for each t in T.*

Proof. Because S_t is increasing in t on T, S_t is a nonempty sublattice of R^n for each t in T. For x in R^n and z in R^n, define $g(x, z) = -\sum_{i=1}^{n} |x_i - z_i|$. Because $g(x, z)$ is continuous and each S_t is nonempty and compact, the maximum of $g(x, z)$ over z in S_t is attained for each (x, t) in $R^n \times T$. Let $f(x, t) = \max_{z \in S_t} g(x, z)$ for (x, t) in $R^n \times T$. By part (b) of Theorem 2.4.5 and part (d) of Example 2.2.5, $\{(x, z, t) : t \in T, z \in S_t, x \in R^n\}$ is a sublattice of $R^{2n} \times T$. By Corollary 2.6.1, $g(x, z)$ is supermodular in (x, z, t) on $R^{2n} \times T$ as in part (g) of Example 2.6.2. Therefore, $f(x, t)$ is supermodular on $R^n \times T$ by Theorem 2.7.6. If t is in T and x is in S_t, then $f(x, t) = g(x, x) = 0$. If t is in T and x is in R^n but not in S_t, then $f(x, t) = g(x, z) < 0$ for some z in S_t. Therefore, $\text{argmax}_{x \in R^n} f(x, t) = S_t$ for each t in T. □

Corollary 2.8.4 applies Theorem 2.8.9 to show that, with a few regularity conditions, the sets of optimal solutions for *any* parameterized collection of maximization problems with increasing optimal solutions are identical with the sets of optimal solutions for another parameterized collection of maximization problems where the objective function is supermodular.

Corollary 2.8.4. *If T is a chain, S_t is a compact subset of R^n for each t in T, $g(x, t)$ is upper semicontinuous in x on R^n for each t in T, and* $\text{argmax}_{x \in S_t} g(x, t)$ *is increasing in t on T, then there exists a supermodular function $f(x, t)$ on $R^n \times T$ such that* $\text{argmax}_{x \in S_t} f(x, t) = \text{argmax}_{x \in R^n} f(x, t) = \text{argmax}_{x \in S_t} g(x, t)$ *for each t in T.*

Corollary 2.8.5 combines the results of Corollary 2.8.4 and Theorem 2.8.6 to show that, with a few regularity conditions, the sets of optimal solutions for a parameterized collection of maximization problems satisfying the ordinal sufficient conditions of Theorem 2.8.6 based on quasisupermodularity and the single crossing property are identical with the sets of optimal solutions for another parameterized collection of maximization problems where the objective function is supermodular. While Example 2.6.6 shows that the collection of quasisupermodular functions is strictly richer than the collection of all strictly increasing transformations of supermodular functions, the result of Corollary 2.8.5 limits the impact of that added generality because the sets of optimal solutions for a parameterized collection of maximization problems with a quasisupermodular objective function can be generated as the sets of optimal solutions for a parameterized collection of maximization problems with a supermodular objective function.

Corollary 2.8.5. *If T is a chain, S_t is a compact subset of R^n for each t in T, S_t is increasing in t on T, $g(x, t)$ is upper semicontinuous and quasisupermodular in x on R^n for each t in T, and $g(x, t)$ satisfies the single crossing property in (x, t) on $R^n \times T$, then there exists a supermodular function $f(x, t)$ on $R^n \times T$ such that* $\text{argmax}_{x \in S_t} f(x, t) = \text{argmax}_{x \in R^n} f(x, t) = \text{argmax}_{x \in S_t} g(x, t)$ *for each t in T.*

Theorem 2.8.10 and Theorem 2.8.11 state a converse of Theorem 2.8.1, characterizing supermodular functions and functions with increasing differences in terms of the existence of increasing optimal selections for the maximization of any linear perturbation of the objective function under consideration. Milgrom and Shannon [1994] establish versions of Theorem 2.8.10 and Theorem 2.8.11 based on increasing optimal solutions rather than increasing optimal selections. A version of Theorem 2.8.10 based on increasing optimal solutions rather than increasing optimal selections follows from Theorem 2.6.5 together with the necessary ordinal condition of Theorem 2.8.13. A version of Theorem 2.8.11 based on increasing optimal solutions rather than increasing optimal selections follows from Theorem 2.6.6 together with the necessary ordinal condition of Theorem 2.8.14.

Theorem 2.8.10. *Suppose that X is a sublattice of R^n and $f(x)$ is a real-valued function on X. If there exists an increasing selection with respect to t in $\{1, 2\}$ from* $\operatorname{argmax}_{x \in S_t}(f(x) + p \cdot x)$ *for each p in R^n and all subsets S_1 and S_2 of X with $S_1 \sqsubseteq S_2$ and* $\operatorname{argmax}_{x \in S_t}(f(x) + p \cdot x)$ *nonempty for $t = 1, 2$, then $f(x)$ is supermodular on X.*

Proof. Suppose that $f(x)$ is not supermodular on X, so there exist unordered x' and x'' in X with $f(x' \vee x'') - f(x'') < f(x') - f(x' \wedge x'')$. Pick p in R^n such that

$$f(x' \vee x'') - f(x'') < p \cdot (x'' - x' \vee x'') = p \cdot (x' \wedge x'' - x')$$
$$< f(x') - f(x' \wedge x'').$$

Let $S_1 = \{x' \wedge x'', x'\}$ and $S_2 = \{x'', x' \vee x''\}$, so $S_1 \sqsubseteq S_2$. Then $\operatorname{argmax}_{x \in S_1}(f(x) + p \cdot x) = \{x'\}$ and $\operatorname{argmax}_{x \in S_2}(f(x) + p \cdot x) = \{x''\}$, and there does not exist an increasing selection from $\operatorname{argmax}_{x \in S_t}(f(x) + p \cdot x)$ for $t = 1, 2$. □

Theorem 2.8.11. *Suppose that X is a subset of R^n, T is a partially ordered set, and $f(x, t)$ is a real-valued function on $X \times T$. If there exists an increasing selection with respect to t from* $\operatorname{argmax}_{x \in X'}(f(x, t) + p \cdot x)$ *for any two distinct t in T with* $\operatorname{argmax}_{x \in X'}(f(x, t) + p \cdot x)$ *nonempty, for each p in R^n, and for each sublattice X' of X, then $f(x, t)$ has increasing differences in (x, t) on $X \times T$.*

Proof. Suppose that $f(x, t)$ does not have increasing differences in (x, t) on $X \times T$, so there exist $x' < x''$ in X and $t' \prec t''$ in T with $f(x'', t'') - f(x', t'') < f(x'', t') - f(x', t')$. Let $X' = \{x', x''\}$. Pick p in R^n such that

$$f(x'', t'') - f(x', t'') < p \cdot (x' - x'') < f(x'', t') - f(x', t').$$

Then $\operatorname{argmax}_{x \in X'}(f(x, t') + p \cdot x) = \{x''\}$ and $\operatorname{argmax}_{x \in X'}(f(x, t'') + p \cdot x) = \{x'\}$, and there does not exist an increasing selection from $\operatorname{argmax}_{x \in X'}(f(x, t) + p \cdot x)$ for $t = t', t''$. □

The necessity result of Theorem 2.8.12 corresponds to the necessity results of Theorem 2.8.10 and Theorem 2.8.11, but weakens the conditions of the former results by only considering optimization with respect to x over the entire domain for x rather than over various subsets of the domain for x. These weakened conditions are achieved by expanding the collection of perturbation functions from all linear functions to all separable functions. The proof applies Theorem 2.6.3 to show that a separable perturbation function can implicitly confine the domain in a suitable manner as is done explicitly in Theorem 2.8.10 and Theorem 2.8.11. The statement of Theorem 2.8.12 is based on increasing optimal solutions rather than increasing optimal selections. However, the result would hold if the assumption of increasing optimal solutions is replaced by the assumption of increasing optimal selections together with the assumption that the set of maxima of each perturbed objective function is a sublattice. (See Example 2.8.6.) The result of Theorem 2.8.12 was prompted by a suggestion from Paul Milgrom.

Theorem 2.8.12. *If $X_1, \ldots, X_n$ and T are chains, $X = \times_{i=1}^n X_i$, S is a sublattice of $X \times T$, $S_t = \{x : (x, t) \in S\}$ is the section of S at t in T, $f(x, t)$ is a real-valued function on S, there exists a real number γ such that $f(x'', t) - f(x', t) < \gamma/2$ for all t in T, x' in S_t, and x'' in S_t, and $\text{argmax}_{x \in S_t}(f(x, t) + w(x))$ is increasing in t on $\{t : t \in T, \text{argmax}_{x \in S_t} f(x, t)$ is nonempty$\}$ for each separable function $w(x)$ on X, then $f(x, t)$ is supermodular in (x, t) on S.*

Proof. Suppose that $f(x, t)$ is not supermodular in (x, t) on S. By Theorem 2.6.3, there exist unordered (x', t') and (x'', t'') in S such that, assuming without loss of generality that $t' \wedge t'' = t' \preceq t'' = t' \vee t''$,

$$f(x' \vee x'', t'') + f(x' \wedge x'', t') < f(x', t') + f(x'', t'') \tag{2.8.3}$$

and

$$S \cap ((\times_{i=1}^n \{x'_i, x''_i\}) \times \{t', t''\}) = \{(x', t'), (x'', t''), (x' \vee x'', t''), (x' \wedge x'', t')\}.$$

The hypothesis that $\text{argmax}_{x \in S_t}(f(x, t) + w(x))$ is increasing in t for each separable function $w(x)$ implies that $\text{argmax}_{x \in S_t}(f(x, t) + w(x))$ is a sublattice of X for each t in T and each separable function $w(x)$. Theorem 2.7.4 implies that $f(x, t)$ is supermodular in x on S_t for each t in T. The supermodularity of $f(x, t)$ in x together with (2.8.3) implies that $t' \neq t''$, and so $t' \prec t''$.

Because (x', t') and (x'', t'') are unordered, there exists i' with $x''_{i'} \prec x'_{i'}$. Define the separable function $w(x) = \sum_{i=1}^n w_i(x_i)$ on X by

$$w_i(x_i) = \begin{cases} -\gamma & \text{if } x_i \text{ is in } X_i \setminus \{x'_i, x''_i\} \\ 0 & \text{if } x_i = x'_i \\ (f(x', t') - f(x'', t'') + f(x' \vee x'', t'') & \\ \qquad -f(x' \wedge x'', t'))/2 & \text{if } i = i' \text{ and } x_i = x''_i \\ 0 & \text{if } i \neq i' \text{ and } x_i = x''_i. \end{cases}$$

If x is in $S_{t'} \setminus (\times_{i=1}^n \{x'_i, x''_i\})$, then

$$\begin{aligned} f(x, t') + w(x) &\leq f(x, t') - \gamma + \max\{w_{i'}(x''_{i'}), 0\} \\ &< f(x', t') = f(x', t') + w(x'). \end{aligned}$$

If x is in $S_{t''} \setminus (\times_{i=1}^{n}\{x'_i, x''_i\})$, then

$$\begin{aligned} f(x, t'') + w(x) &\leq f(x, t'') - \gamma + \max\{w_{i'}(x''_{i'}), 0\} \\ &< f(x'', t'') + w(x''). \end{aligned}$$

Thus,

$$\text{argmax}_{x \in S_{t'}}(f(x, t') + w(x)) \subseteq S_{t'} \cap (\times_{i=1}^{n}\{x'_i, x''_i\}) = \{x', x' \wedge x''\}$$

and

$$\text{argmax}_{x \in S_{t''}}(f(x, t'') + w(x)) \subseteq S_{t''} \cap (\times_{i=1}^{n}\{x'_i, x''_i\}) = \{x'', x' \vee x''\}.$$

Furthermore, using (2.8.3),

$$\begin{aligned} &f(x' \wedge x'', t') + w(x' \wedge x'') \\ &\quad = f(x', t') - (f(x', t') + f(x'', t'') - f(x' \vee x'', t'') - f(x' \wedge x'', t'))/2 \\ &\quad < f(x', t') = f(x', t') + w(x') \end{aligned}$$

and

$$\begin{aligned} &f(x' \vee x'', t'') + w(x' \vee x'') \\ &\quad = f(x' \vee x'', t'') \\ &\quad < f(x' \vee x'', t'') + (f(x', t') + f(x'', t'') \\ &\qquad - f(x' \vee x'', t'') - f(x' \wedge x'', t'))/2 \\ &\quad = f(x'', t'') + w(x''). \end{aligned}$$

Consequently,

$$\text{argmax}_{x \in S_{t'}}(f(x, t') + w(x)) = \{x'\}$$

and

$$\text{argmax}_{x \in S_{t''}}(f(x, t'') + w(x)) = \{x''\}.$$

This together with $x''_{i'} \prec x'_{i'}$ contradicts $\text{argmax}_{x \in S_{t'}}(f(x, t') + w(x)) \sqsubseteq \text{argmax}_{x \in S_{t''}}(f(x, t'') + w(x))$ (and it also contradicts the existence of increasing optimal selections from $\text{argmax}_{x \in S_t}(f(x, t) + w(x))$ for t in $\{t', t''\}$). □

Example 2.8.5 shows that the result of Theorem 2.8.12, giving necessary conditions for supermodularity, would not hold if the set of perturbation functions consists of all linear functions rather than all separable functions. Example 2.8.5 also shows that the necessity result of Theorem 2.8.11 would not hold if the objective function therein is maximized only over its entire domain rather than over an arbitrary sublattice of its domain.

Example 2.8.5. Let $X = \{0, 1, 2\}$, $T = \{0, 1\}$, $f(x, t) = 1$ for x in $\{0, 2\}$ and t in T, $f(1, 0) = 0$, and $f(1, 1) = \delta$ where $\delta < 1$. For any t in T,

$$\text{argmax}_{x \in X}(f(x, t) + px) = \begin{cases} \{2\} & \text{if } p > 0 \\ \{0, 2\} & \text{if } p = 0 \\ \{0\} & \text{if } p < 0. \end{cases}$$

Thus, $\text{argmax}_{x \in X}(f(x, t) + px)$ is increasing in t for each p in R^1. For any δ, $f(2, 1) - f(2, 0) = f(0, 1) - f(0, 0) = 0$ and $f(1, 1) - f(1, 0) = \delta$, so $f(2, 1) - f(2, 0) = 0 < \delta = f(1, 1) - f(1, 0)$ if $0 < \delta$ and $f(1, 1) - f(1, 0) = \delta < 0 = f(0, 1) - f(0, 0)$ if $\delta < 0$. Therefore, for any $\delta \neq 0$, the function $f(x, t)$ is not supermodular in (x, t) and does not have increasing differences in (x, t) on $X \times T$.

Example 2.8.6 shows that Theorem 2.8.12 would not hold if its hypotheses were weakened by assuming increasing optimal selections instead of increasing optimal solutions, without also requiring that $\text{argmax}_{x \in S_t}(f(x, t) + w(x))$ is a sublattice of X for each t in T and for each separable function $w(x)$ on X.

Example 2.8.6. Suppose that $X_1, \ldots, X_n$ are chains, $X = \times_{i=1}^n X_i$, $g(x)$ is any bounded real-valued function on X that is not supermodular on X, $T = \{1\}$, $S = X \times T$, and $f(x, t) = g(x)$ for (x, t) in S. Then there is an increasing optimal selection from $\text{argmax}_{x \in S_t}(f(x, t) + w(x))$ for t in T for each separable function $w(x)$ on X, and all the hypotheses of Theorem 2.8.12 hold except for $\text{argmax}_{x \in S_1}(f(x, 1) + w(x)) \sqsubseteq \text{argmax}_{x \in S_1}(f(x, 1) + w(x))$; that is, $\text{argmax}_{x \in S_1}(f(x, 1) + w(x))$ need not be a sublattice of X for each separable function $w(x)$ on X. However, $f(x, t)$ is not supermodular in (x, t).

Theorem 2.8.13 and Theorem 2.8.14, from Milgrom and Shannon [1994], show that the ordinal sufficient conditions of Theorem 2.8.6 are necessary for increasing optimal solutions, thereby providing a characterization of those properties of the objective function that yield increasing optimal solutions.

Theorem 2.8.13. *Suppose that X is a lattice and $f(x)$ is a function from X into some chain. If $\text{argmax}_{x \in S_t} f(x)$ is increasing in t on $\{t : t \in T$, $\text{argmax}_{x \in S_t} f(x)$ is nonempty$\}$ for any partially ordered set T and for any collection of subsets S_t of X with S_t increasing in t on T, then $f(x)$ is quasi-supermodular in x on X.*

Proof. Pick any x' and x'' in X. Let $T = \{1, 2\}$, $S_1 = \{x', x' \wedge x''\}$, and $S_2 = \{x'', x' \vee x''\}$, so $S_1 \sqsubseteq S_2$. If $f(x' \wedge x'') \preceq f(x')$, then x' is in $\text{argmax}_{x \in S_1} f(x)$ and so $x' \vee x''$ is in $\text{argmax}_{x \in S_2} f(x)$ and $f(x'') \preceq f(x' \vee x'')$ because $\text{argmax}_{x \in S_1} f(x) \sqsubseteq \text{argmax}_{x \in S_2} f(x)$. If $f(x' \wedge x'') \prec f(x')$, then $\text{argmax}_{x \in S_1} f(x) = \{x'\}$ and so $\text{argmax}_{x \in S_2} f(x) = \{x' \vee x''\}$ and $f(x'') \prec f(x' \vee x'')$ because $\text{argmax}_{x \in S_1} f(x) \sqsubseteq \text{argmax}_{x \in S_2} f(x)$. Thus, $f(x)$ is quasisupermodular in x on X. □

Example 2.8.7 shows that quasisupermodularity is not necessary for the existence of increasing optimal selections, and so the statement of Theorem 2.8.13 cannot be generalized to involve the existence of increasing optimal selections instead of increasing optimal solutions.

Example 2.8.7. Suppose that $X = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, $f(0, 0) = 0$, and $f(0, 1) = f(1, 0) = f(1, 1) = 1$. The function $f(x)$ is not quasisupermodular on X because $f((0, 1) \wedge (1, 0)) = f(0, 0) = 0 < 1 = f(0, 1)$ and $f(1, 0) = 1 = f(1, 1) = f((0, 1) \vee (1, 0))$. Let T be a partially ordered set and let S_t be any collection of subsets of X with S_t increasing in t on T. For each t in T, let $x(t)$ be the greatest element of S_t (which exists by Lemma 2.2.1). The form of $f(x)$ for the present example implies that $x(t)$ is in $\text{argmax}_{x \in S_t} f(x)$ for each t in T, so $x(t)$ is an optimal selection. By Theorem 2.4.3, $x(t)$ is increasing in t, and so $x(t)$ is an increasing optimal selection. Thus, even though $f(x)$ is not quasisupermodular on X, an increasing optimal selection exists for the parameterized collection of optimization problems of maximizing $f(x)$ subject to x in S_t, where S_t is any increasing collection of subsets of X. (Observe the case with $T = \{1, 2\}$, $S_1 = \{(0, 0), (0, 1)\}$, and $S_2 = \{(1, 0), (1, 1)\}$, so $S_1 \sqsubseteq S_2$, $\text{argmax}_{x \in S_1} f(x) = \{(0, 1)\}$ and $\text{argmax}_{x \in S_2} f(x) = \{(1, 0), (1, 1)\}$. Here, $x(1) = (0, 1) < (1, 1) = x(2)$, so $x(t)$ is an increasing optimal selection but $\text{argmax}_{x \in S_t} f(x)$ is not increasing in t on T.)

Theorem 2.8.14. *Suppose that X is a lattice, T is a partially ordered set, and $f(x, t)$ is a function from $X \times T$ into some chain. If* $\text{argmax}_{x \in X'} f(x, t)$ *is increasing in t on $\{t : t \in T, \text{argmax}_{x \in X'} f(x, t)$ is nonempty$\}$ for each sublattice X' of X, then $f(x, t)$ satisfies the single crossing property in (x, t) on $X \times T$.*

Proof. Pick any x' and x'' in X with $x' \prec x''$ and t' and t'' in T with $t' \prec t''$. Let $X' = \{x', x''\}$. If $f(x', t') \preceq f(x'', t')$, then x'' is in $\text{argmax}_{x \in X'} f(x, t')$ and so x'' is in $\text{argmax}_{x \in X'} f(x, t'')$ and $f(x', t'') \preceq f(x'', t'')$ because $\text{argmax}_{x \in X'} f(x, t') \sqsubseteq \text{argmax}_{x \in X'} f(x, t'')$. If $f(x', t') \prec f(x'', t')$, then $\text{argmax}_{x \in X'} f(x, t') = \{x''\}$ and so $\text{argmax}_{x \in X'} f(x, t'') = \{x''\}$ and $f(x', t'') \prec f(x'', t'')$ because $\text{argmax}_{x \in X'} f(x, t') \sqsubseteq \text{argmax}_{x \in X'} f(x, t'')$. Thus, $f(x, t)$ satisfies the single crossing property in (x, t) on $X \times T$. □

Example 2.8.8 shows that the single crossing property is not necessary for the existence of an increasing optimal selection, and so the statement of Theorem 2.8.14 cannot be generalized to involve the existence of increasing optimal selections instead of increasing optimal solutions.

Example 2.8.8. Suppose that $X = T = \{0, 1\}$, $f(0, 0) = 0$, and $f(0, 1) = f(1, 0) = f(1, 1) = 1$. The function $f(x, t)$ does not have the single crossing property on $X \times T$ because $f(0, 0) = 0 < 1 = f(1, 0)$ and $f(0, 1) = 1 = f(1, 1)$. If X' is a nonempty sublattice of X, then $x(t) = 1$ for each t in T is an increasing optimal selection from $\text{argmax}_{x \in X'} f(x, t)$ if 1 is in X' and $x(t) = 0$ for each t in T is an increasing optimal selection from $\text{argmax}_{x \in X'} f(x, t)$ otherwise. Thus, there exists an increasing optimal selection from $\text{argmax}_{x \in X'} f(x, t)$ for each nonempty sublattice X' of X, even though $f(x, t)$ does not have the single crossing property. (Observe that $\text{argmax}_{x \in X} f(x, 0) = \{1\}$ and $\text{argmax}_{x \in X} f(x, 1) = \{0, 1\}$, so $\text{argmax}_{x \in X} f(x, t)$ is not increasing in t on T.)

2.9 Complementarity Equivalences

Previous sections in this chapter considered several different properties that are logically or technically related to the concept of complementarity. This section summarizes equivalences and implications that exist between these properties.

For convenience of exposition at this level of presentation, mention of various technical details is omitted in this section. This looseness and impreciseness of the statements below distinguish the articulation in the present section from that in the other nonintroductory sections of this monograph. Precise statements can be gleaned from the results cited in the three paragraphs following the listing of the properties.

Suppose that $x = (x_1, \ldots, x_n)$ is an n-vector of consumption levels of n products $i = 1, \ldots, n$ and $f(x)$ is the real-valued utility of x. Consider the following ten properties (A–J) for the n products. (Some of these properties can be stated more generally. Refer to the earlier results cited after the listing of these properties.) Relationships already established among properties (A–J) are discussed following their listing. The first six properties (A–F) are equivalent. The next two properties (G–H) are equivalent and are strictly weaker than the first six properties. Furthermore, properties (G–H) imply both property (I) and property (J). Property (A) is taken as the definition of complementarity in this monograph. Properties (A), (C), and (H) offer the most straightforward complementarity interpretations. Properties (D), (E), and (F) are stronger perturbed versions of properties (G), (H), and (J), respectively.

(A) **Increasing differences**. The utility function $f(x)$ has increasing differences.
(B) **Supermodularity**. The utility function $f(x)$ is supermodular.
(C) **Price monotonicity**. If $p = (p_1, \dots, p_n)$ is a price vector for the acquisition of the n products and the consumption levels are chosen to maximize $f(x) - p \cdot x$, the utility minus the acquisition cost, then these optimal consumption levels are decreasing in p.
(D) **Perturbed utility function is quasisupermodular**. Each linear perturbation of the utility function is quasisupermodular.
(E) **Optimal consumption level monotonicity with perturbed utility function**. If the consumption levels of any subset of the products are taken as a parameter or if the set of all feasible solutions is taken as a parameter, then the optimal consumption levels for any separable or linear perturbation of the utility function are an increasing function of the parameter.
(F) **Optimal solutions for perturbed utility function form a sublattice**. The collection of optimal solutions for each linear or separable perturbation of the utility function is a sublattice.
(G) **Quasisupermodularity**. The utility function $f(x)$ is quasisupermodular.
(H) **Optimal consumption level monotonicity**. If the consumption levels of any subset of the products are taken as a parameter or if the set of all feasible solutions is taken as a parameter, then the optimal consumption levels are an increasing function of the parameter.
(I) **Equivalent supermodular utility function exists**. Taking the consumption level of any particular product as a parameter, there exists a supermodular utility function such that the collection of optimal consumption levels for the other $n - 1$ products is the same as for the original utility function $f(x)$.
(J) **Optimal solutions form sublattice**. The collection of all optimal consumption levels is a sublattice.

The following earlier results establish the equivalence of properties (A–F). By Corollary 2.6.1, (A) implies (B). By Theorem 2.6.1, (B) implies (A). By Corollary 2.8.2, (B) implies (C). By Theorem 2.8.8, (C) implies (B). By Theorem 2.6.5, (B) and (D) are equivalent. By Theorem 2.8.2 (together with part (b) of Lemma 2.6.1 and Theorem 2.6.4), (B) implies (E). By Theorem 2.8.10 and Theorem 2.8.12, (E) implies (B). By Theorem 2.7.1 (together with part (b) of Lemma 2.6.1 and Theorem 2.6.4), (B) implies (F). By Theorem 2.7.3 and Theorem 2.7.4, (F) implies (B).

By part (b) of Lemma 2.6.5, (B) implies (G). By Example 2.6.6, (G) is a strictly weaker property than (any strictly increasing transformation of) (B). (See also Example 2.6.7.) By Theorem 2.8.6 (together with part (c) of Lemma 2.6.5), (G) implies (H). By Theorem 2.8.13, (H) implies (G).

By Corollary 2.8.4, (H) implies (I). By Theorem 2.7.2, (G) implies (J).

3

Optimal Decision Models

3.1 Introduction

This chapter applies the results of Chapter 2 to optimal decision models. The primary focus is on establishing increasing optimal solutions and complementarity properties. The various sections of this chapter can be read independently, except that Section 3.7 should be read before Subsection 3.8.2 and Subsection 3.9.1 should be read before Section 3.10. The sections are organized as follows.

Section 3.2 examines matching problems, where workers of different types are to be assigned among multiple firms. Conditions are given for optimal matchings to be increasing or ordered with respect to the qualities of the workers and the efficiencies of the firms. An expanded version of the model includes operational decisions within the firm.

Section 3.3 analyzes five versions of a general model for a firm engaged in manufacturing and marketing operations. Sufficient and necessary conditions are given on the demand functions and the cost functions for complementarities to hold and for optimal decisions to increase with a parameter. Various specific examples are included.

Section 3.4 considers parametric properties in the transportation problem and in the two-stage transshipment problem. For the transportation problem, complementarity and substitutability properties are established among the capacities at the factories and the demands of the customers (as well as among the operating factories and the customers in the market) and the optimal dual variables are monotone with respect to problem parameters. Examples show that seemingly natural extensions of these complementarity and substitutability properties do not generally hold for relationships involving warehouses (located between the factories and the customers) in the two-stage transshipment problem.

Section 3.5 clarifies the role of supermodularity for increasing optimal solutions and planning horizon results in dynamic economic lot size production models. The analysis is based on properties derived for minimum cost paths in acyclic networks.

Section 3.6 considers sensitivity analysis in a deterministic multiperiod multiproduct production planning problem. Conditions are given for optimal cumulative production levels to increase with cumulative demands and with bounds on inventory, and for those parameters to be complementary.

Section 3.7 is concerned with structural and qualitative properties of two related network problems, the minimum cut problem and the maximum closure problem. Complementarities and increasing optimal solutions are analyzed for these problems. An application is given for the selection problem, where a firm must determine a subset of available activities in which to engage. The structure of the sets of optimal solutions for the minimum cut problem or for the maximum closure problem yields a characterization of the structure of certain sublattices and the structure of the sets of maxima of supermodular functions over certain sublattices.

Section 3.8 gives conditions under which a firm engaged in activities over multiple time periods finds it optimal to myopically determine its decisions in each time period; that is, to optimize for each individual period without taking into account the impact of present decisions on subsequent time periods. Such optimal myopic decisions increase from period to period. An application involves a dynamic version of the selection problem.

Section 3.9 considers Markov decision processes. Conditions are presented under which optimal decisions increase with the state in each period and the components of the state are complementary. Applications are given for advertising, pricing, and the maintenance of an unreliable system. The results rely on properties of certain property-inducing stochastic transformations, which are also developed in this section.

Section 3.10 gives conditions for optimal inventory decisions in a multiperiod stochastic inventory problem to vary monotonically with parametric changes in the problem's costs and demand distributions. To facilitate the analysis for these results, conditions are found for parameterized stochastic transformations to preserve supermodularity.

3.2 Matching

This section considers the optimal matching of different types of workers to multiple firms. Conditions are given under which some or all optimal matchings are increasing or ordered.

Consider m firms $j = 1, \dots, m$, with the firms drawing on a labor market for n types of workers $i = 1, \dots, n$. Each of the m firms requires exactly one worker of each of the n types. No worker can be hired by more that one firm. A **matching** assigns exactly one worker of each of the n types to each of the

m firms with no worker assigned to more than one firm. The quality of each worker of each type is represented by a real number. A matching is denoted by the specification of the quality of each worker of each type assigned to each firm. (This representation of a matching may not distinguish between all distinct assignments of workers to firms if the qualities of different workers of any given type are not all distinct, but that ambiguity is inconsequential in the present context.)

Let X_i be the set consisting of each quality of each available type i worker, so there is a one-to-one relationship between the elements of X_i and the type i workers in the labor market. (Because different type i workers may have the same quality, the elements of X_i may not be distinct.) There are finitely many workers of each type in the labor market. Assume that there are at least m workers of each type i in the labor market, so $|X_i| \geq m$ for each i and some matching exists. The labor market is **tight** if there are exactly m workers of each type; that is, if $|X_i| = m$ for each i. In a tight labor market, each matching has each worker in the labor market being hired by some firm. The labor market is **loose** if there are at least m workers of each given type and quality for which at least one worker of that type and quality exists; that is, if z' is in X_i for some worker type i, then $|\{z : z = z', z \in X_i\}| \geq m$. In a loose labor market, the labor market is large enough so that the hiring opportunities for any firm (that is, the qualities of workers available for hiring) are not impacted by the hiring decisions of the other firms. If the labor market is not loose, then the hiring opportunities for any firm may be impacted by the hiring decisions of the other firms. Tight and loose labor markets are extreme cases with respect to the availability of workers in the labor market.

Let x_i denote the quality of a type i worker. Let $x = (x_1, \ldots, x_n)$ denote the vector of qualities of n workers, with exactly one worker of each type and with x_i being the quality of the type i worker. If exactly one worker of each type i is assigned to each firm j, let x_i^j denote the quality of the type i worker assigned to firm j, let $x^j = (x_1^j, \ldots, x_n^j)$ be the vector of qualities of the n workers assigned to each firm j, and let $x^1, \ldots, x^m$ denote the assignments of workers to all m firms. An assignment of workers $x^1, \ldots, x^m$ is a **matching** if $\{x_i^1, \ldots, x_i^m\}$ is a subset of X_i for $i = 1, \ldots, n$; that is, if exactly one worker of each type is assigned to each firm with no worker assigned to more than one firm. A matching $x^1, \ldots, x^m$ is **increasing** if $x^j \leq x^{j+1}$ for $j = 1, \ldots, m-1$; that is, if x^j is increasing in j. In a tight labor market, an increasing matching assigns the j^{th} best worker (for some resolution of ties in quality) of each type to firm $n - j + 1$ and the representation for an increasing matching is unique. (However, because the qualities of different workers of the same type need not be distinct, more than one actual assignment of workers may correspond to the unique representation for an increasing matching in a tight labor market.) A matching $x^1, \ldots, x^m$ is **ordered** if the vectors $x^{j'}$ and $x^{j''}$ of worker assignments to any two distinct firms j' and j'' are ordered; that is, if either $x^{j'} \leq x^{j''}$ or $x^{j''} \leq x^{j'}$. In a tight labor market, an ordered matching assigns the j^{th} best worker (for some resolution of ties in quality) of each type to the same firm and the representation for any ordered matching is a permutation (with respect to the m sets of worker qualities assigned to the m firms) of the unique representation for an increasing matching.

If firm j hires workers with qualities $x = (x_1, \ldots, x_n)$, then firm j receives a profit $f(x, j)$. The dependence of the profit function $f(x, j)$ on j may reflect differences among firms in factors such as technology, capital, organization, and management. Generally, the firms may be considered as ordered by the index j according to their respective efficiency, with subsequent hypotheses clarifying this interpretation. The firms are **homogeneous** if $f(x, j)$ does not depend on j.

A matching that maximizes the sum of the profits for all the firms is **optimal**. The problem of finding an optimal matching, the **matching problem**, is then to

$$\begin{aligned} &\text{maximize } \textstyle\sum_{j=1}^{m} f(x^j, j) \\ &\text{subject to } \{x_i^1, \ldots, x_i^m\} \subseteq X_i \text{ for } i = 1, \ldots, n. \end{aligned} \tag{3.2.1}$$

Because at least one matching exists (since $|X_i| \geq m$ for each i) and there are finitely many matchings (since X_i is finite for each i), an optimal matching exists.

While the present model is stated in terms of each firm hiring exactly one worker of each type, the model implicitly permits a firm to hire more than one worker of a given type or no worker of a given type. To permit a firm to hire more than one worker of a given type, the model could replicate that type as multiple distinct worker types. (This requires a modification to (3.2.1) because here workers modeled as distinct types would come from the same labor pool.) Permitting a firm to hire no worker of a given type is accomplished by adding m artificial workers to each X_i, where these m workers have a quality strictly less than any other quality in X_i and where hiring any of these artificial workers impacts a firm's profit function the same as if the firm hired no worker of that type.

Becker [1973], attributing the proof to William Brock, shows that each optimal matching is ordered if $n = 2$, the firms are homogeneous, the labor market is tight, and $f(x, j)$ is strictly supermodular in x. For arbitrary n, homogeneous firms, and a tight labor market, Lin [1992] shows that each ordered matching is optimal if $f(x, j)$ is supermodular in x and that each optimal matching is ordered if $f(x, j)$ is strictly supermodular in x. Kremer [1993] analyzes firms with arbitrary n, a loose labor market, and a specialized profit function based on a Cobb-Douglas production function with all exponents equal to 1, and shows that each optimal matching is ordered. The present section extends those results for firms that need not be homogeneous, labor markets that need not be tight or loose, and some more detailed modeling of the firms' operations.

The key hypothesis in subsequent results has to do with the supermodularity of the profit function $f(x, j)$ in (x, j) on $(\times_{i=1}^{n} X_i) \times \{1, \ldots, m\}$. By Theorem 2.6.1 and Corollary 2.6.1, this supermodularity is equivalent to the profit function $f(x, j)$ having increasing differences with respect to the quality of each pair of distinct worker types and increasing differences with respect to the quality of each worker type and the index j of the firm. The former condition reflects complementarities between different worker types, so that the additional net profit for any firm from increasing the quality of any one worker type in the firm increases with the qualities of the other worker types in the firm. The latter condition reflects complementarities between each worker type and the firm's efficiency, so that the additional net profit for any firm from increasing the quality of any one worker type in the firm increases with the firm's efficiency.

If the labor market is loose, then the hiring decisions of each firm in an optimal matching may be determined by maximizing its individual profit function over the entire labor market without regard for the hiring decisions of the other firms. In this case, the matching problem (3.2.1) simplifies to solving

$$\text{maximize } f(x^j, j) \text{ subject to } x^j \in \times_{i=1}^{n} X_i \tag{3.2.2}$$

for each firm $j = 1, \ldots, m$. A matching $x^1, \ldots, x^m$ is optimal if and only if each x^j is an optimal solution for (3.2.2).

Theorem 3.2.1 shows that if the labor market is loose and $f(x, j)$ is supermodular in (x, j), then the optimal hiring decisions for the m firms are increasing in j (with respect to the induced set ordering $\sqsubseteq$) and an increasing optimal matching exists. Theorem 3.2.1 follows from Theorem 2.8.2 and either part (a) or part (b) of Theorem 2.8.3.

Theorem 3.2.1. *If the labor market is loose and $f(x, j)$ is supermodular in (x, j) on $(\times_{i=1}^{n} X_i) \times \{1, \ldots, m\}$, then the set of optimal hiring decisions* $\text{argmax}_{x \in \times_{i=1}^{n} X_i} f(x, j)$ *for each firm j is increasing (with respect to the induced set ordering $\sqsubseteq$) in j and there exists an increasing optimal matching.*

Theorem 3.2.2 provides a converse for Theorem 3.2.1. That is, if the labor market is loose and the optimal hiring decisions for the m firms are increasing in j (with respect to the induced set ordering $\sqsubseteq$) given any wage rates for the n types of workers where the workers' wages are incorporated within $f(x, j)$, then $f(x, j)$ is supermodular in (x, j). Theorem 3.2.2 follows from Theorem 2.8.12.

Theorem 3.2.2. *If the labor market is loose, $g(x, j)$ is the profit to firm j not including wages paid to its workers where firm j hires workers with a vector of qualities x, $w_i(x_i)$ is the wage paid by any firm to a worker of type i with quality x_i in X_i, the profit function for firm j is $f(x, j) = g(x, j) + \sum_{i=1}^{n} w_i(x_i)$, and the set of optimal hiring decisions* $\text{argmax}_{x \in \times_{i=1}^{n} X_i} f(x, j)$ *for each firm j is increasing (with respect to the induced set ordering $\sqsubseteq$) in j for each vector of wage functions $w_1(x_1), \ldots, w_n(x_n)$, then $g(x, j)$ is supermodular in (x, j) on $(\times_{i=1}^{n} X_i) \times \{1, \ldots, m\}$.*

Theorem 3.2.3 gives conditions for there to exist an increasing optimal matching, without the hypothesis of Theorem 3.2.1 that the labor market is loose. The conclusion about optimal matchings is somewhat stronger in Theorem 3.2.1, and, unlike the more complicated proof of Theorem 3.2.3, the proof for Theorem 3.2.1 follows directly from general results on increasing optimal solutions for supermodular functions. When the labor market is tight, the conditions of Theorem 3.2.3 imply that each increasing matching is optimal.

Theorem 3.2.3. *If $f(x, j)$ is supermodular in (x, j) on $(\times_{i=1}^{n} X_i) \times \{1, \ldots, m\}$, then there exists an increasing optimal matching of workers to firms. If, in addition, the labor market is tight, then each increasing matching is optimal.*

Proof. Suppose that there does not exist any increasing optimal matching. Then for any optimal matching $x^1, \ldots, x^m$, there must be some i and some $j \geq 2$ with $x_i^j < \max_{1 \leq k < j} x_i^k$. Pick an optimal matching $\bar{x}^1, \ldots, \bar{x}^m$, a firm j', and a worker type i' such that $\max_{1 \leq k < j} \bar{x}_i^k \leq \bar{x}_i^j$ for all j and i either with $j > j'$ or with $j = j'$ and $i > i'$, $\bar{x}_{i'}^{j'} < \max_{1 \leq k < j'} \bar{x}_{i'}^k$, and such that if $x^1, \ldots, x^m$ is any other optimal matching then either $x_i^j < \max_{1 \leq k < j} x_i^k$ for some $j > j'$ and some i or $x_i^{j'} < \max_{1 \leq k < j'} x_i^k$ for some $i \geq i'$. (That is, the optimal matching $\bar{x}^1, \ldots, \bar{x}^m$ lexicographically minimizes over all optimal matchings $x^1, \ldots, x^m$ the lexicographically greatest (j, i) with $x_i^j < \max_{1 \leq k < j} x_i^k$, and the pair (j', i') corresponds to this solution.) Pick j'' with $1 \leq j'' < j'$ and $\bar{x}_{i'}^{j'} < \max_{1 \leq k < j'} \bar{x}_{i'}^k = \bar{x}_{i'}^{j''}$. Here, $1 \leq j'' < j' \leq m$ and $1 \leq i' \leq n$.

Define a matching $\hat{x}^1, \ldots, \hat{x}^m$ such that $\hat{x}^j = \bar{x}^j$ for each j with $j \neq j'$ and $j \neq j''$, $\hat{x}^{j'} = \bar{x}^{j'} \vee \bar{x}^{j''}$, and $\hat{x}^{j''} = \bar{x}^{j'} \wedge \bar{x}^{j''}$. By the choice of $\bar{x}^1, \ldots, \bar{x}^m$, j', i', and j'', it follows that $\max_{1 \leq k < j} \hat{x}_i^k = \max_{1 \leq k < j} \bar{x}_i^k \leq \bar{x}_i^j = \hat{x}_i^j$ for all j and i either with $j > j'$ or with $j = j'$ and $i > i'$, $\max_{1 \le k < j'} \hat{x}^k_{i'} \le \max_{1 \le k < j'} \bar{x}^k_{i'} = \bar{x}^{j''}_{i'} = \hat{x}^{j'}_{i'}$, and $\hat{x}^1, \ldots, \hat{x}^m$ is not an optimal matching. Therefore,

$$\begin{aligned} 0 &< \textstyle\sum_{j=1}^m f(\bar{x}^j, j) - \sum_{j=1}^m f(\hat{x}^j, j) \\ &= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\hat{x}^{j'}, j') - f(\hat{x}^{j''}, j'') \qquad (3.2.3) \\ &= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\bar{x}^{j'} \vee \bar{x}^{j''}, j' \vee j'') - f(\bar{x}^{j'} \wedge \bar{x}^{j''}, j' \wedge j''). \end{aligned}$$

But (3.2.3) contradicts the supermodularity of $f(x, j)$ in (x, j), so there exists an increasing optimal matching.

When the labor market is tight, each increasing matching has the same representation and yields the same value for $\sum_{j=1}^m f(x^j, j)$ and so each increasing matching is optimal. □

Theorem 3.2.4 gives conditions for each optimal matching to be ordered. (Regarding the hypotheses of Theorem 3.2.4, note that $f(x, j)$ is supermodular in (x, j) and strictly supermodular in x if and only if $f(x, j)$ has increasing differences in (x, j) and is strictly supermodular in x. The sufficiency of this statement follows from Theorem 2.6.2, and necessity follows from Theorem 2.6.1.)

Theorem 3.2.4. *If $f(x, j)$ is supermodular in (x, j) on $(\times_{i=1}^n X_i) \times \{1, \ldots, m\}$ and is strictly supermodular in x on $\times_{i=1}^n X_i$ for each j, then each optimal matching is ordered.*

Proof. Suppose that there exists some optimal matching that is not ordered. Let $\bar{x}^1, \ldots, \bar{x}^m$ be such an optimal matching. Then there exist firms j' and j'' such that $j' < j''$ and $\bar{x}^{j'}$ and $\bar{x}^{j''}$ are unordered. Define the matching $\hat{x}^1, \ldots, \hat{x}^m$ such that $\hat{x}^j = \bar{x}^j$ for all j with $j \ne j'$ and $j \ne j''$, $\hat{x}^{j''} = \bar{x}^{j'} \vee \bar{x}^{j''}$, and $\hat{x}^{j'} = \bar{x}^{j'} \wedge \bar{x}^{j''}$. Because $\bar{x}^1, \ldots, \bar{x}^m$ is an optimal matching, $\hat{x}^1, \ldots, \hat{x}^m$ is a matching, and $f(x, j)$ is supermodular in (x, j) and strictly supermodular in x,

$$\begin{aligned} 0 &\le \textstyle\sum_{j=1}^m f(\bar{x}^j, j) - \sum_{j=1}^m f(\hat{x}^j, j) \\ &= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\hat{x}^{j'}, j') - f(\hat{x}^{j''}, j'') \\ &= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\bar{x}^{j'} \wedge \bar{x}^{j''}, j') - f(\bar{x}^{j'} \vee \bar{x}^{j''}, j'') \\ &\le f(\bar{x}^{j'}, j'') + f(\bar{x}^{j''}, j'') - f(\bar{x}^{j'} \wedge \bar{x}^{j''}, j'') - f(\bar{x}^{j'} \vee \bar{x}^{j''}, j'') \\ &< 0, \end{aligned}$$

which is a contradiction. Therefore, each optimal matching is ordered. □

Theorem 3.2.5 gives conditions for each optimal matching to be increasing. When these conditions hold and the labor market is tight, the results of Theorem 3.2.5 and Theorem 3.2.3 establish that a matching is optimal if and only if it is increasing.

Theorem 3.2.5. *If $f(x, j)$ is strictly supermodular in (x, j) on $(\times_{i=1}^n X_i) \times \{1, \ldots, m\}$, then each optimal matching is increasing.*

Proof. Pick any optimal matching $\bar{x}^1, \ldots, \bar{x}^m$. By Theorem 3.2.4, $\bar{x}^1, \ldots, \bar{x}^m$ is ordered and so each pair from $\bar{x}^1, \ldots, \bar{x}^m$ is ordered. Suppose that $\bar{x}^1, \ldots, \bar{x}^m$ is not increasing, so there exist firms j' and j'' with $j' < j''$ and $\bar{x}^{j''} < \bar{x}^{j'}$. Define the matching $\hat{x}^1, \ldots, \hat{x}^m$ such that $\hat{x}^j = \bar{x}^j$ for each j with $j \neq j'$ and $j \neq j''$, $\hat{x}^{j'} = \bar{x}^{j''}$, and $\hat{x}^{j''} = \bar{x}^{j'}$. Because $\bar{x}^1, \ldots, \bar{x}^m$ is an optimal matching, $\hat{x}^1, \ldots, \hat{x}^m$ is a matching, and $f(x, j)$ is strictly supermodular in (x, j),

$$\begin{aligned}
0 &\leq \textstyle\sum_{j=1}^m f(\bar{x}^j, j) - \sum_{j=1}^m f(\hat{x}^j, j) \\
&= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\hat{x}^{j'}, j') - f(\hat{x}^{j''}, j'') \\
&= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\bar{x}^{j''}, j') - f(\bar{x}^{j'}, j'') \\
&= f(\bar{x}^{j'}, j') + f(\bar{x}^{j''}, j'') - f(\bar{x}^{j'} \wedge \bar{x}^{j''}, j' \wedge j'') - f(\bar{x}^{j'} \vee \bar{x}^{j''}, j' \vee j'') \\
&< 0,
\end{aligned}$$

which is a contradiction. Therefore, each optimal matching is increasing. □

Now generalize the statement of a matching problem to include operational decision variables for each firm in addition to the hiring assignments of workers to each firm. After a vector x of n workers is assigned to any firm j, the firm determines a k-vector of operational decision variables z from a subset $S_{x,j}$ of R^k so as to maximize a profit function $g(z, x, j)$. Assume that each maximization problem for each firm has an optimal solution. Given an assignment x of workers to firm j, the profit for firm j after optimizing with respect to its operational decision variables is

$$f(x, j) = \max_{z \in S_{x,j}} g(z, x, j). \tag{3.2.4}$$

With this construction of $f(x, j)$, the **matching problem** for optimally assigning workers to firms is the same as that defined previously in this section. Define

$$S = \{(z, x, j) : x \in \times_{i=1}^n X_i, j \in \{1, \ldots, m\}, z \in S_{x,j}\}.$$

Theorem 3.2.6 gives conditions for there to exist an increasing optimal matching and for the firms' optimal operational decision variables to increase (with respect to the induced set ordering $\sqsubseteq$) with j. The key assumption in Theorem 3.2.6 is the supermodularity of $g(z, x, j)$ in (z, x, j). By Theorem

2.6.1 and Corollary 2.6.1, this assumption is equivalent to complementarities between each pair of components of (z, x, j). Complementarities among pairs of components of (x, j) are discussed above Theorem 3.2.1. Here, there are additional complementarities between each component of the operational decision and the quality of each worker type, between each component of the operational decision and the efficiency of the firm, and between each pair of components of the operational decision. (A similar interpretation holds for the strict supermodularity assumption in Theorem 3.2.7.)

Theorem 3.2.6. *Suppose that S is a sublattice of R^{k+n+1} and $g(z, x, j)$ is supermodular in (z, x, j) on S.*
(a) *The profit function $f(x, j)$ is supermodular in (x, j) on $(\times_{i=1}^{n} X_i) \times \{1, \dots, m\}$.*
(b) *There exists an increasing optimal matching of workers to firms.*
(c) *Given any increasing matching $x^1, \dots, x^m$ of workers to firms (including an increasing optimal matching as exists by part (b)), the set of optimal operational decisions* $\mathrm{argmax}_{z \in S_{x^j, j}} g(z, x^j, j)$ *for each firm j is increasing in j.*

Proof. Part (a) follows from (3.2.4), Theorem 2.7.6, and the supermodularity of $g(z, x, j)$ on S.

Part (b) follows from part (a) and Theorem 3.2.3.

Part (c) follows from Theorem 2.8.2 and the supermodularity of $g(z, x, j)$ on S. □

Theorem 3.2.7 gives conditions for each optimal matching to be increasing.

Theorem 3.2.7. *Suppose that S is a sublattice of R^{k+n+1} and $g(z, x, j)$ is strictly supermodular in (z, x, j) on S.*
(a) *The profit function $f(x, j)$ is strictly supermodular in (x, j) on $(\times_{i=1}^{n} X_i) \times \{1, \dots, m\}$.*
(b) *Each optimal matching of workers to firms is increasing.*
(c) *Given any optimal matching $x^1, \dots, x^m$ of workers to firms, the set of optimal operational decisions* $\mathrm{argmax}_{z \in S_{x^j, j}} g(z, x^j, j)$ *for each firm j is increasing in j.*

Proof. Part (a) follows from (3.2.4), Theorem 2.7.7, and the strict supermodularity of $g(z, x, j)$ on S.

Part (b) follows from part (a) and Theorem 3.2.5.

Part (c) follows from part (b) and from part (c) of Theorem 3.2.6. □

3.3 Comparative Statics of the Firm

This section studies the role of complementarity and supermodularity for the decision making of a firm engaged in manufacturing and marketing operations. Subsection 3.3.1 formulates a general model of the firm. Subsection 3.3.2

considers five versions of the general model of Subsection 3.3.1 and gives sufficient conditions for broad classes of demand functions and cost functions so that complementarities hold and optimal decisions are monotone with a parameter. The sufficient conditions are applied to diverse examples, including complex integrated models of manufacturing and marketing; competitive pricing and quality selection with substitute products; production expansion and momentum over time with the acquisition of knowledge and technology; and pricing and quality selection, where price and quality are substitute attributes in the market. Subsection 3.3.3 establishes that key hypotheses for sufficiency in Subsection 3.3.2 are also necessary for the monotonicity of optimal decisions. This section is based on material in Topkis [1995a].

3.3.1 Model of the Firm

This subsection formulates a model of a profit-maximizing firm that produces and markets a single product in one time period. (As shown in more detail in Subsection 3.3.2, this model may nevertheless incorporate multiple products by treating the single product as an aggregate approximation of multiple products and by including the number or variety of products as a variable (as in Bagwell and Ramey [1994] and Milgrom and Roberts [1990b]). Multiple time periods can be included by letting time be a parameter (as in Milgrom, Qian, and Roberts [1991] and Milgrom and Roberts [1990b]). Furthermore, the influence of competitors may be reflected in the parameters (as in Milgrom and Roberts [1990a] and Topkis [1979]); see Subsection 4.4.1.) The environment in which the firm operates is determined by parameters, which may be variable parameters or constant parameters. The **variable parameters** are represented as an m-vector t included in a subset T of R^m. One is interested in how different values of the variable parameter t, determining different instances of the firm's decision problem, may impact the optimal decisions and profits of the firm. The **constant parameters** also affect the firm's environment, but these are considered to be fixed for all instances of the firm's decision problem and their impact on the firm is not part of the analysis. For each instance of the decision problem determined by a particular value for the variable parameter t, the firm selects a decision variable x, where x is an n-vector and is constrained to be in a subset X of R^n. Depending on the particular model under consideration, the components of the parameters and the decision variables may include such factors as technology, unit production cost, quality, number or variety of products, knowledge, more detailed aspects of the manufacturing and design processes, advertising, market size, prices of competitors' substitute products, and time. The price of the product is p, which is included in a subset P of R^1 and which, depending on the model, may be either a decision variable, a variable parameter, or a constant parameter. Because of the special role of price in the model, p is treated distinctly from t and x even if price is, respectively, a variable parameter or a decision variable. The product demand is $\mu(p, x, t)$, which may depend on the price p, the decision variable x, and the variable parameter t. For a level of production z, there is a production cost $c(z, x, t)$, which may depend on the decision variable x and the variable parameter t as well as on the level of production. It is assumed that the firm's production equals the demand $\mu(p, x, t)$ with all demand satisfied by current production, so the production cost is $c(\mu(p, x, t), x, t)$ and the revenue is $p\mu(p, x, t)$. There are also costs $k(x, t)$, which may depend on the decision variable x and the variable parameter t but not on the level of production nor on the product demand. Given price p, decision variable x, and variable parameter t, the firm's profit is

$$\Pi(p, x, t) = p\mu(p, x, t) - c(\mu(p, x, t), x, t) - k(x, t), \tag{3.3.1}$$

which is the difference between the firm's revenue $p\mu(p, x, t)$ and the sum of its production cost $c(\mu(p, x, t), x, t)$ and its other costs $k(x, t)$. (A convention here is to express profit functions and their components with arguments being the decision variables and the variable parameters, where the constant parameters are not explicitly included among the arguments. In Subsection 3.3.2, this leads to some variations from the notation introduced above.) For given values of the variable parameters, the firm selects its decision variables to maximize profit. The form of the firm's parametric decision problem depends on whether price is a decision variable, a variable parameter, or a constant parameter.

3.3.2 Sufficient Conditions

The model summarized in the profit function of (3.3.1) includes somewhat more generality than can readily be shown to exhibit complementarities and monotone optimal solutions without some limitations. Following are five more specific models that consider different specializations of (3.3.1) and analyze the model under hypotheses that yield complementarities and monotone optimal solutions. The next two paragraphs summarize generally the similarities and distinctions between the five subsequent models, as well as noting several conventions and hypotheses.

One may consider the model of (3.3.1) as having three degrees of generality. These are price p being a variable (either a decision variable or a variable parameter), the production cost being nonlinear and depending on all decision variables and variable parameters, and the demand function depending on all decision variables and variable parameters. The general model is tractable with any two of these three degrees of generality, but not with all three. (Using the equivalence between supermodularity and increasing differences established in Theorem 2.6.1 and Corollary 2.6.1, it can be seen that general conditions for the supermodularity of $\Pi(p, x, t)$ in (p, x, t) do not exist when all three degrees of generality are present.) The following five specific models reflect this limitation on the generality of the model. The first model, described in Example 3.3.1 and analyzed in Theorem 3.3.1, limits the first degree of generality and requires that price be a constant parameter, while the demand function and the nonlinear production cost function may depend on all variables. The second model, described in Example 3.3.3 and analyzed in Theorem 3.3.2, is out of step with the other models of this subsection in that the profit function $\Pi(p, x, t)$ is not supermodular and various other anomalies are present. This model requires that the production cost function be linear (where the production cost coefficient may be either a decision variable or a variable parameter), while price is a decision variable and the demand function may depend on all variables. The third model, described in Example 3.3.5 and analyzed in Theorem 3.3.3, limits the second degree of generality and requires that the production cost function be linear with the production cost coefficient being a constant parameter, while price may be either a decision variable or a variable parameter and the demand function may depend on all variables. The fourth model, described in Example 3.3.7 and analyzed in Theorem 3.3.4, limits the third degree of generality and requires that the demand function depend on no variable other than price, while price may be either a decision variable or a variable parameter and the nonlinear production cost function may depend on all variables. The fifth model, described in Example 3.3.9 and analyzed in Theorem 3.3.5, limits but does not entirely eliminate both the second and third degrees of generality so that both the demand function and the nonlinear production cost function may depend on an additional variable with that dependence being limited, while price may be either a decision variable or a variable parameter. Where price is a decision variable (in the models of Example 3.3.3, Example 3.3.5, Example 3.3.7, and Example 3.3.9), the model may represent a monopoly industry or a firm in monopolistic competition. Where price is taken by the firm as given so price is either a constant parameter or a variable parameter (in the models of Example 3.3.1, Example 3.3.5, Example 3.3.7, and Example 3.3.9), the model may represent a competitive firm or a regulated monopoly.

In order to simplify the presentation and to focus on essential properties, it is convenient here to specify certain conventions and regularity conditions that, without further mention, are implicitly made where appropriate throughout this subsection. All domains of functions, sets of parameters, and constraint sets for optimization problems are nonempty. An optimal solution exists for each maximization problem. Where one has an increasing collection of sets and is interested in selecting a particular solution from each set such that these selections are increasing, it is assumed that each set is compact so each set has a greatest (least) element and the greatest (least) element is increasing by Theorem 2.4.3. (This last assumption holds for sets of optimal decisions if each set of feasible decision variables is nonempty and compact and if each profit function is upper semicontinuous in the decision variable.) To facilitate more succinct statements, the same hypotheses are stated for applications of Theorem 2.7.6 (the preservation of supermodularity under the maximization operation) and Theorem 2.8.2 (complementarity implying increasing optimal solutions) even though Theorem 2.7.6 alone requires that the set of variable parameters is a sublattice and that the profit function is supermodular in the variable parameter. To avoid thereby requiring unnecessarily strong assumptions for statements involving the application of Theorem 2.8.2, it is taken as a convention that assumptions made for such statements are only required to hold for each subset of variable parameters that is a chain.

The model presented in Example 3.3.1 and analyzed in Theorem 3.3.1 is a version of the Subsection 3.3.1 model that treats price as a constant parameter and otherwise has general demand and production costs.

Example 3.3.1. A firm operates according to the general model of Subsection 3.3.1, with the following refinements. The price p is a constant parameter. The demand for the firm's product is $\mu(x, t)$. The firm's profit is

$$\Pi(x, t) = p\mu(x, t) - c(\mu(x, t), x, t) - k(x, t). \tag{3.3.2}$$

Theorem 3.3.1. *Consider the model of Example* 3.3.1 *with the profit function* $\Pi(x, t)$ *of* (3.3.2)*. Suppose that X and T are sublattices, $\mu(x, t)$ is increasing and supermodular in (x, t), $pz - c(z, x, t)$ is increasing in z for each (x, t), $c(z, x, t)$ is concave in z for each (x, t), $c(z, x, t)$ is submodular in (z, x, t), and $k(x, t)$ is submodular in (x, t). Then,*
(a) $\Pi(x, t)$ *is supermodular in* (x, t)*;*
(b) $\max_{x\in X} \Pi(x, t)$ *is supermodular in* t*; and*
(c) $\operatorname{argmax}_{x\in X} \Pi(x, t)$ *is increasing in* (t, X)*.*

Proof. By the hypotheses, $pz - c(z, x, t)$ is increasing and convex in z for each (x, t) and is supermodular in (z, x, t). Part (a) then follows from (3.3.2), Lemma 2.6.4, and part (b) of Lemma 2.6.1. Part (b) and part (c) follow from part (a) and from Theorem 2.7.6 and Theorem 2.8.2, respectively. □

For the case where the production cost is linear so $c(z, x, t) = c(x, t)z$, the hypotheses on $c(z, x, t)$ in Theorem 3.3.1 reduce to $p \geq c(x, t)$ for each (x, t) and $c(x, t)$ is decreasing and submodular in (x, t).

Example 3.3.2, analyzed in Corollary 3.3.1, is a special case of Example 3.3.1. This example corresponds to a version of the manufacturing model of Milgrom and Roberts [1990b], with a significant difference being that the model of Milgrom and Roberts [1990b] treats the price as a decision variable while this example restricts price to being a constant parameter. Example 3.3.2 incorporates features of the more market-oriented model of Bagwell and Ramey [1994], which also treats price as a decision variable. (Example 3.3.4 considers a version of this model that has price as a decision variable.) Furthermore, the present model has a concave production cost changing with time and depending on a technology variable, while the models of Bagwell and Ramey [1994] and Milgrom and Roberts [1990b] (and Example 3.3.4) have a linear production cost. Other differences between the model and assumptions of this example and those of Milgrom and Roberts [1990b] are as in the differences for Example 3.3.4 noted in the paragraph preceding Corollary 3.3.2. See Subsection 4.4.2.

Example 3.3.2. A firm produces multiple products in a single period, but the model aggregates properties like price and demand so that the model resembles that of a single product firm. The time period under consideration is τ. The price p is a constant parameter. There are η products. The total number of improvements added to all products in the period is q. The firm spends α on advertising for its products. The products are exposed to a market of size σ. The base demand for the firm's product is $\nu(\eta, q, \alpha, \sigma, \tau)$.

The ultimate demand for the firm's product may be less than $\nu(\eta, q, \alpha, \sigma, \tau)$ because of cancellations due to delays in filling orders. Assume that the firm immediately responds to orders with accurate delay estimates and that any order cancellations are also immediate, so no production-related costs are associated with cancelled orders. Let θ denote the level of technology used in the manufacturing process. If γ is the number of setups per period and r is the probability that a batch is defective, then $\omega(\eta, \theta, \gamma, r)$ is the time required to manufacture the demand requested in a processed order. Furthermore, there are delays a and b for processing orders and for delivering finished products, respectively. There is a shrinkage function $\delta(a+\omega(\eta, \theta, \gamma, r)+b, \tau)$, such that the ultimate demand after all cancellations is $\nu(\eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, \theta, \gamma, r) + b, \tau)$.

The cost per unit for repairing defective batches is a constant parameter ρ. There is a cost h per unit of inventory held per period, where h is a constant parameter and the interval for holding inventory is the time between successive setups. The cost of producing z units of product is $\lambda(z, \theta, \tau)$. The total expected cost associated with producing an amount z is thus $\lambda(z, \theta, \tau) + \rho r z + hz/\gamma$.

Each setup incurs a direct cost s and a wastage cost w, resulting in a cost $\gamma(s + w)$ per period. Each product improvement incurs a design cost d and extra setup costs e, resulting in a cost $q(d + e)$ per period. The capital cost associated with the technology-related variables is $\kappa(\eta, \theta, r, a, b, s, w, d, e, \tau)$. In addition to the advertising cost α, there is a cost $\psi(\sigma, \tau)$ for servicing a market of size σ in period τ. Thus the total cost not associated with the level of production is $\gamma(s+w)+q(d+e)+\kappa(\eta, \theta, r, a, b, s, w, d, e, \tau)+\alpha+\psi(\sigma, \tau)$.

For variables $(\eta, q, \alpha, \sigma, \theta, \gamma, r, a, b, s, w, d, e, \tau)$, the firm's profit is

$$
\begin{aligned}
\Pi(\eta, q, \alpha, \sigma, \theta, \gamma, r, a, b, s, w, d, e, \tau) \\
= p\nu(\eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, \theta, \gamma, r) + b, \tau) \\
- \lambda(\nu(\eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, \theta, \gamma, r) + b, \tau), \theta, \tau) \\
- (\rho r + h/\gamma)\nu(\eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, \theta, \gamma, r) + b, \tau) \\
- \gamma(s + w) - q(d + e) - \kappa(\eta, \theta, r, a, b, s, w, d, e, \tau) - \alpha - \psi(\sigma, \tau).
\end{aligned}
\tag{3.3.3}
$$

In stating Corollary 3.3.1, it is convenient to replace some of the variables in (3.3.3) with other variables that are their negatives. Let the collection of all variables be $(\eta, q, \alpha, \sigma, \theta, \gamma, -r, -a, -b, -s, -w, -d, -e, \tau)$. Let the vector of decision variables x be any nonempty subset of these variables not including τ, and let the vector of variable parameters t consist of all variables not included in x. The decision variable x is constrained to be in a set X, and the variable parameter t is taken from a set T.

Corollary 3.3.1. *Consider the model of Example 3.3.2. Suppose that X and T are sublattices; $\nu(\eta, q, \alpha, \sigma, \tau)$ is nonnegative, increasing, and supermodular in $(\eta, q, \alpha, \sigma, \tau)$; $\delta(z, \tau)$ is nonnegative, decreasing and convex in z for each τ, increasing in τ for each z, and submodular in (z, τ); $\omega(\eta, \theta, \gamma, r)$ is decreasing and submodular in $(\eta, \theta, \gamma, -r)$; $pz - \lambda(z, \theta, \tau) - (\rho r + h/\gamma)z$ is increasing in z for each $(\theta, \gamma, r, \tau)$; $\lambda(z, \theta, \tau)$ is concave in z for each (θ, τ) and submodular in (z, θ, τ); $\kappa(\eta, \theta, r, a, b, s, w, d, e, \tau)$ is submodular in $(\eta, \theta, -r, -a, -b, -s, -w, -d, -e, \tau)$; and $\psi(\sigma, \tau)$ is submodular in (σ, τ). Then,*

(a) $\Pi(x, t)$ *is supermodular in* (x, t)*;*

(b) $\max_{x \in X} \Pi(x, t)$ *is supermodular in* t*; and*

(c) $\operatorname{argmax}_{x \in X} \Pi(x, t)$ *is increasing in* (t, X).

Proof. The proof proceeds by observing correspondences between the terms in (3.3.3) and (3.3.2) and showing that the conditions of Theorem 3.3.1 hold.

The conditions on $c(z, x, t) = \lambda(z, \theta, \tau) + (\rho r + h/\gamma)z$ and on $k(x, t) = \gamma(s + w) + q(d + e) + \kappa(\eta, \theta, r, a, b, s, w, d, e, \tau) + \alpha + \psi(\sigma, \tau)$ in Theorem 3.3.1 follow directly from hypotheses. To complete the proof, it suffices to show that $\mu(x, t) = \nu(\eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, \theta, \gamma, r) + b, \tau)$ is increasing and supermodular in $(\eta, q, \alpha, \sigma, \theta, \gamma, -r, -a, -b, \tau)$.

By hypotheses, $\nu(\eta, q, \alpha, \sigma, \tau)$ is nonnegative, increasing, and supermodular in $(\eta, q, \alpha, \sigma, \tau)$ and $\delta(a + \omega(\eta, \theta, \gamma, r) + b, \tau)$ is nonnegative and increasing in $(\eta, \theta, \gamma, -r, -a, -b, \tau)$. Because the product of two nonnegative, increasing, supermodular functions is supermodular by Corollary 2.6.3, it suffices to show that $\delta(a+\omega(\eta, \theta, \gamma, r)+b, \tau)$ is supermodular in $(\eta, \theta, \gamma, -r, -a, -b, \tau)$. This last condition holds by Lemma 2.6.4 because $\delta(z, \tau)$ is supermodular in $(-z, \tau)$ and is increasing and convex in $-z$ for each τ and $(-a-\omega(\eta, \theta, \gamma, r)-b)$ is increasing and supermodular in $(\eta, \theta, \gamma, -r, -a, -b, \tau)$. □

The model presented in Example 3.3.3 and analyzed in Theorem 3.3.2 is a version of the Subsection 3.3.1 model that treats price as a decision variable, has a general demand function, and has linear production costs with the unit production cost being a variable (either a decision variable or a variable parameter).

Example 3.3.3. A firm operates according to the general model of Subsection 3.3.1, with the following refinements. The price p is a decision variable. The cost of producing an amount z of the product is cz, where c is a variable, either a decision variable or a variable parameter, and must be included in a subset C of R^1. Assume that $P \cap (c, \infty)$ is nonempty for each c in C. Let $-P = \{-p : p \in P\}$ and $-C = \{-c : c \in C\}$. The decision variable x does not include the cost c if c is a decision variable. The variable parameter t in T does not include the cost c if c is a variable parameter. The costs $k(c, x, t)$ depend on c, x, and t but not on the level of production. The firm's profit is

$$\Pi(p, c, x, t) = p\mu(p, x, t) - c\mu(p, x, t) - k(c, x, t). \tag{3.3.4}$$

Theorem 3.3.2. *Consider the model of Example* 3.3.3 *with the profit function* $\Pi(p, c, x, t)$ *of* (3.3.4). *Suppose that X and T are sublattices, $\mu(p, x, t)$ is positive and increasing in (x, t) and log-supermodular in $(-p, x, t)$, and $k(c, x, t)$ is submodular in $(-c, x, t)$. Then,*

(a) $\max_{p\in P} \Pi(p, c, x, t)$ *is supermodular in* $(-c, x, t)$*;*

(b) $\text{argmax}_{-p\in -P} \Pi(p, c, x, t)$ *is increasing in* $(-c, x, t, -P)$*;*

(c) $\max_{p\in P, x\in X} \Pi(p, c, x, t)$ *and* $\max_{p\in P, c\in C, x\in X} \Pi(p, c, x, t)$ *are supermodular in* $(-c, t)$ *and in* t*, respectively;*

(d) $\mathrm{argmax}_{x\in X}(\max_{p\in P}\Pi(p, c, x, t))$ *and* $\mathrm{argmax}_{-c\in -C, x\in X}(\max_{p\in P}\Pi(p, c, x, t))$ *are increasing in* $(-c, t, X)$ *and in* $(t, -C, X)$*, respectively; and*
(e) $\mathrm{argmax}_{-p\in -P, x\in X}\Pi(p, c, x, t)$ *and* $\mathrm{argmax}_{-p\in -P, -c\in -C, x\in X}\Pi(p, c, x, t)$ *both have greatest and least elements for each* $(c, t, -P, X)$ *and for each* $(t, -P, -C, X)$*, respectively, and these are increasing in* $(-c, t, X)$ *and in* $(t, -C, X)$*, respectively.*

Proof. Because $\mu(p, x, t)$ is positive, in maximizing over p one need only consider p with $p > c$. Because $\log(w)$ is increasing,

$$\begin{aligned}
&\max_{p\in P}\Pi(p, c, x, t) + k(c, x, t)\\
&\quad = \max_{p\in P}(p - c)\mu(p, x, t)\\
&\quad = \exp(\log(\max_{\{-p: -p < -c, -p\in -P\}}(p - c)\mu(p, x, t)))\\
&\quad = \exp(\max_{\{-p: -p < -c, -p\in -P\}}(\log((p - c)\mu(p, x, t)))).
\end{aligned} \tag{3.3.5}$$

Because $\mu(p, x, t)$ is log-supermodular in $(-p, x, t)$ and $\log(w)$ is concave for $w > 0$, $\log((p - c)\mu(p, x, t)) = \log(p - c) + \log(\mu(p, x, t))$ is supermodular in $(-p, -c, x, t)$ by part (b) of Lemma 2.6.2 and part (b) of Lemma 2.6.1. Then by Theorem 2.7.6, $\max_{\{-p: -p < -c, -p\in -P\}}(\log((p - c)\mu(p, x, t)))$ is supermodular in $(-c, x, t)$. Also, $\max_{\{-p: -p < -c, -p\in -P\}}(\log((p - c)\mu(p, x, t)))$ is increasing in $(-c, x, t)$, since $\mu(p, x, t)$ is positive and increasing in (x, t). Because $\exp(z)$ is increasing and convex, Lemma 2.6.4 and (3.3.5) imply that $\max_{p\in P}\Pi(p, c, x, t) + k(c, x, t)$ is supermodular in $(-c, x, t)$. Then part (a) follows from the submodularity of $k(c, x, t)$ in $(-c, x, t)$.

Part (b) follows from Theorem 2.8.2, the supermodularity of $\log((p - c)\mu(p, x, t))$ in $(-p, -c, x, t)$, and

$$\begin{aligned}
\mathrm{argmax}_{-p\in -P}\Pi(p, c, x, t) &= \mathrm{argmax}_{\{-p: -p < -c, -p\in -P\}}((p - c)\mu(p, x, t))\\
&= \mathrm{argmax}_{\{-p: -p < -c, -p\in -P\}}(\log((p - c)\mu(p, x, t))).
\end{aligned}$$

Part (c) and part (d) follow from part (a) and from Theorem 2.7.6 and Theorem 2.8.2, respectively.

Part (e) follows from part (b) and part (d). □

The model of Example 3.3.3 generalizes that of Example 3.3.1 by having price as a decision variable rather than a constant parameter, but, otherwise, Theorem 3.3.2 makes stronger assumptions than Theorem 3.3.1. The earlier assumption of a concave production cost function that is supermodular in all variables is replaced by the assumption of a linear production cost, where the coefficient is a variable. Here the demand function is log-supermodular in $(-p, x, t)$ rather than supermodular in (x, t).

Theorem 3.3.2 exhibits several apparent idiosyncrasies, discussed in the following paragraphs. These have to do with the direction of the change in price with respect to the variable parameters; the proof establishing the supermodularity of $\max_{p \in P} \Pi(p, c, x, t)$ in $(-c, x, t)$ in part (a); the hypothesis that the demand function $\mu(p, x, t)$ is log-supermodular in $(-p, x, t)$; the form of the monotonicity statement in part (e); no need for the demand function to be decreasing in price; and the lack of a corresponding version of Theorem 3.3.2 for which p is a variable parameter instead of a decision variable. This model, its hypotheses, and its proof are different in character from others in this subsection because no strictly increasing transformation of the profit function is supermodular in all variables.

In Theorem 3.3.2, as with the hypotheses for each other model in this subsection, the demand function is increasing in the non-price variables. The impact on profit of that hypothesis tends to make higher prices relatively more attractive for higher values of the non-price variable parameters. Subject to other hypotheses, this seems to suggest that price would increase with the non-price variable parameters affecting demand, and that is indeed the case in each of the three subsequent models. In Theorem 3.3.2, on the contrary, price decreases with those non-price variable parameters that may affect demand (that is, t). This helps explain why the hypotheses and proof of this result are somewhat different in character from other results in this subsection.

Part (a) of Theorem 3.3.2 states that applying the maximization operation with respect to p to the profit function of (3.3.4) yields a supermodular function. Typically, the proof of such a statement would proceed by first showing that the profit function is supermodular in all variables and then invoking the result of Theorem 2.7.6 that supermodularity is preserved under the maximization operation. This is not a viable approach here because the profit function $\Pi(p, c, x, t)$ need not be supermodular in $(-p, -c, x, t)$ or in $(p, -c, x, t)$ under the hypotheses of Theorem 3.3.2.

Theorem 3.3.2 assumes that the demand function is a log-supermodular function of its arguments, while the results for other models in this subsection for other versions of (3.3.1) only assume that the demand function is supermodular. Milgrom and Roberts [1990a] use log-supermodularity and the log transformation in a Bertrand oligopoly model, involving a special case of Theorem 3.3.2 with $k(c, x, t) = 0$ and the decision variable x omitted. See Subsection 2.6.4 and Subsection 4.4.1.

Results on increasing optimal solutions are typically phrased in terms of the sets of optima increasing with a parameter, where the ordering on the sets is the induced set ordering $\sqsubseteq$, as in Theorem 2.8.1 and Theorem 2.8.2. Then increasing optimal selections would follow if regularity conditions hold as in Theorem 2.8.3. However, part (e) of Theorem 3.3.2 only states that the greatest and least elements of $\text{argmax}_{-p\in -P, x\in X}\,\Pi(p, c, x, t)$ and $\text{argmax}_{-p\in -P, -c\in -C, x\in X}\,\Pi(p, c, x, t)$ exist and are increasing in $(-c, t, X)$ and in $(t, -C, X)$, respectively, rather than that the sets of optima are increasing. This weakened monotonicity statement stems from the two-stage approach to optimization in Theorem 3.3.2, with first optimizing out p and then optimizing over x or over $(-c, x)$. This two-stage approach corresponds to that of Milgrom and Roberts [1990b]. (See Example 3.3.4 below.) Furthermore, viewed in a more abstract setting, the present two-stage approach and the consequent weakened monotonicity statement correspond to a framework given in a 1991 version of Milgrom and Shannon [1994].

Theorem 3.3.2 does not assume that the demand function $\mu(p, x, t)$ is decreasing in p, so the anomalous case of a Giffen good is permitted.

Typically, if a monotonicity result holds with a particular variable being one of the decision variables, then the result would still hold if that variable were instead a variable parameter. (See the paragraph following Theorem 2.8.1.) However, Theorem 2.8.1 does not apply directly for Theorem 3.3.2, and there is the unusual occurrence where Theorem 3.3.2 holds with p a decision variable but a corresponding result for the same model under the same hypotheses would not hold with p a variable parameter instead. To see this, let $P = [0, 99]$, $X = [.5, 50]$, $\mu(p, x, t) = (100 - p)x$, $c = 0$, and $k(c, x, t) = x^2$, so $\Pi(p, c, x, t) = p(100 - p)x - x^2$. Then the conditions of Theorem 3.3.2 hold, since $\mu(p, x, t)$ is increasing in (x, t) and (multiplicatively separable and) log-supermodular in $(-p, x, t)$ and $k(c, x, t)$ is submodular in $(-c, x, t)$. The optimal value of x given p is $p(100 - p)/2$, which is strictly increasing in p for p in $[0, 50]$ and is strictly decreasing in p for p in $[50, 99]$. Thus the optimal value of x is neither increasing nor decreasing with p. This example with $P = \{p\}$, where $0 \le p \le 99$, also indicates that part (e) of Theorem 3.3.2 cannot be extended to include monotonicity with respect to $-P$.

Example 3.3.4, analyzed in Corollary 3.3.2, is a special case of Example 3.3.3.

Example 3.3.4. Consider the model of Example 3.3.2, but with the modifications that price p is a decision variable; the base demand function explicitly depends on price, as $\nu(p, \eta, q, \alpha, \sigma, \tau)$; the cost of producing z units of product is cz, where c is a variable (either a decision variable or a variable parameter); and the technology variable θ is replaced by the marginal production cost c, which is an inverse measure of the level of technology. For variables

$(p, \eta, q, \alpha, \sigma, c, \gamma, r, a, b, s, w, d, e, \tau)$, the firm's profit is

$$\begin{aligned}
&\Pi(p, \eta, q, \alpha, \sigma, c, \gamma, r, a, b, s, w, d, e, \tau)\\
&\quad = p\nu(p, \eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, c, \gamma, r) + b, \tau)\\
&\qquad - (c + \rho r + h/\gamma)\nu(p, \eta, q, \alpha, \sigma, \tau)\delta(a + \omega(\eta, c, \gamma, r) + b, \tau)\\
&\qquad - \gamma(s + w) - q(d + e) - \kappa(\eta, c, r, a, b, s, w, d, e, \tau) - \alpha - \psi(\sigma, \tau).
\end{aligned}$$

Let $(-p, \eta, q, \alpha, \sigma, -c, \gamma, -r, -a, -b, -s, -w, -d, -e, \tau)$ be the collection of all variables, let $-p$ be a decision variable, let the vector of other decision variables x be any subset of the variables not including $-p$ and τ, and let the vector of variable parameters t consist of all variables not included in $(-p, x)$. The decision variable x is constrained to be in a set X and the variable parameter t is taken from a set T.

Following are differences between the model and assumptions of Example 3.3.4 and Corollary 3.3.2 and those of Milgrom and Roberts [1990b]. The number of products η is here taken to be a variable rather than a constant parameter. To facilitate an analysis based on supermodularity with this new variable, the definition of the variable q is changed to the total number of product improvements rather than the average number of improvements per product (which here is q/η). The variables α for advertising and σ for market size are included in the present model, thereby incorporating features of a model of Bagwell and Ramey [1994] that has variables $(p, \eta, \alpha, \sigma, c)$ and a multiplicatively separable (and, hence, log-supermodular) demand function with more refined and specialized demand modeling. Hypotheses on the demand and cost functions are expanded to reflect the new variables. The present results accommodate having any nonempty subset of the variables including price p and not including time τ as the decision variables, rather than all variables except τ. Assumptions of twice differentiability are dropped. The assumption that demand is decreasing in price is dropped. The assumption that the profit function is strictly quasi-concave in p is dropped. The present example adds the new assumption that the base demand function is log-supermodular, strengthening the assumption of Milgrom and Roberts [1990b] that the base demand function is supermodular. This additional assumption is crucial. Indeed, Theorem 3.3.7 demonstrates that log-supermodularity is necessary for monotonicity in the present model. (The statement of the monotonicity result of Milgrom and Roberts [1990b] is not correct under the hypotheses stated therein. A counterexample can be seen with a separable base demand function. See Bushnell and Shepard [1995], Milgrom and Roberts [1995], and Topkis [1995b] for discussions and corrections.)

Corollary 3.3.2. *Consider the model of Example* 3.3.4. *Suppose that X and T are sublattices; $\nu(p, \eta, q, \alpha, \sigma, \tau)$ is positive, increasing in $(\eta, q, \alpha, \sigma, \tau)$, and log-supermodular in $(-p, \eta, q, \alpha, \sigma, \tau)$; $\delta(z, \tau)$ is positive, decreasing and convex in z for each τ, increasing in τ for each z, and submodular in (z, τ); $\omega(\eta, c, \gamma, r)$ is decreasing and submodular in $(\eta, -c, \gamma, -r)$; $\kappa(\eta, c, r, a, b, s, w, d, e, \tau)$ is submodular in $(\eta, -c, -r, -a, -b, -s, -w, -d, -e, \tau)$; and $\psi(\sigma, \tau)$ is submodular in (σ, τ). Then,*
(a) $\max_{p\in P} \Pi(p, c, x, t)$ *is supermodular in* $(-c, x, t)$;
(b) $\text{argmax}_{-p\in -P} \Pi(p, c, x, t)$ *is increasing in* $(-c, x, t, -P)$;
(c) $\max_{p\in P, x\in X} \Pi(p, c, x, t)$ *and* $\max_{p\in P, c\in C, x\in X} \Pi(p, c, x, t)$ *are supermodular in* $(-c, t)$ *and in* t, *respectively;*
(d) $\text{argmax}_{x\in X}(\max_{p\in P} \Pi(p, c, x, t))$ *and* $\text{argmax}_{-c\in -C, x\in X}(\max_{p\in P} \Pi(p, c, x, t))$ *are increasing in* $(-c, t, X)$ *and in* $(t, -C, X)$, *respectively; and*
(e) $\text{argmax}_{-p\in -P, x\in X} \Pi(p, c, x, t)$ *and* $\text{argmax}_{-p\in -P, -c\in -C, x\in X} \Pi(p, c, x, t)$ *both have greatest and least elements for each* $(c, t, -P, X)$ *and for each* $(t, -P, -C, X)$, *respectively, and these are increasing in* $(-c, t, X)$ *and in* $(t, -C, X)$, *respectively.*

Proof. Part (c), part (d), and part (e) follow from part (a) and part (b) as in the proof of Theorem 3.3.2, so it suffices to establish part (a) and part (b).

The hypotheses of Theorem 3.3.2 hold where c' is the marginal production cost variable, $(\eta, q, \alpha, \sigma, \tau)$ are the other non-price variables, the demand function is $\nu(p, \eta, q, \alpha, \sigma, \tau)$, and there are no costs that do not depend on the level of production. Then $\max_{p\in P}(p - c')\nu(p, \eta, q, \alpha, \sigma, \tau)$ is supermodular in $(-c', \eta, q, \alpha, \sigma, \tau)$ by part (a) of Theorem 3.3.2, is increasing in $(-c', \eta, q, \alpha, \sigma, \tau)$ by hypothesis, and is convex in $-c'$ because it is the maximum of a collection of affine functions (in $-c'$). Letting $-c' = -c - \rho r - h/\gamma$, Lemma 2.6.4 implies that $\max_{p\in P}(p - c - \rho r - h/\gamma)\nu(p, \eta, q, \alpha, \sigma, \tau)$ is supermodular in $(-c, \eta, q, \alpha, \sigma, \gamma, -r, \tau)$. Furthermore, $\delta(a + \omega(\eta, c, \gamma, r) + b, \tau)$ is nonnegative, increasing, and supermodular in $(-c, \eta, \gamma, -r, -a, -b, \tau)$ as in the proof of Corollary 3.3.1. Therefore,

$$\begin{aligned}
&\max_{p\in P} \Pi(p, c, x, t)\\
&\quad = (\max_{p\in P}(p - c - \rho r - h/\gamma)\nu(p, \eta, q, \alpha, \sigma, \tau))\delta(a + \omega(\eta, c, \gamma, r) + b, \tau)\\
&\qquad - \gamma(s + w) - q(d + e) - \kappa(\eta, c, r, a, b, s, w, d, e, \tau) - \alpha - \psi(\sigma, \tau)
\end{aligned}$$

is supermodular in $(-c, x, t)$ by Corollary 2.6.3 and part (b) of Lemma 2.6.1, and so part (a) holds. Part (b) holds because part (b) of Theorem 3.3.2

implies that $\text{argmax}_{-p\in -P}(p-(c+\rho r+h/\gamma))\nu(p,\eta,q,\alpha,\sigma,\tau)$ is increasing in $(-(c+\rho r+h/\gamma),\eta,q,\alpha,\sigma,\tau,-P)$. $\square$

The model presented in Example 3.3.5 and analyzed in Theorem 3.3.3 is a version of the Subsection 3.3.1 model that treats price as a variable (either a decision variable or a variable parameter), has a general demand function, and has linear production costs with the unit production cost being a constant parameter.

Example 3.3.5. A firm operates according to the general model of Subsection 3.3.1, with the following refinements. The market price p is a variable, either a decision variable or a variable parameter. The cost of producing an amount z of the product is cz, where c is a constant parameter and P is a subset of $[c,\infty)$. The firm's profit is

$$\Pi(p,x,t) = p\mu(p,x,t) - c\mu(p,x,t) - k(x,t). \tag{3.3.6}$$

Theorem 3.3.3. *Consider the model of Example* 3.3.5 *with the profit function* $\Pi(p,x,t)$ *of* (3.3.6)*. Suppose that X and T are sublattices, $\mu(p,x,t)$ is increasing in (x,t) and supermodular in (p,x,t), and $k(x,t)$ is submodular in (x,t). Then,*
(a) $\Pi(p,x,t)$ *is supermodular in* (p,x,t)*;*
(b) $\max_{x\in X}\Pi(p,x,t)$ *and* $\max_{p\in P,x\in X}\Pi(p,x,t)$ *are supermodular in* (p,t) *and in t, respectively; and*
(c) $\text{argmax}_{x\in X}\Pi(p,x,t)$ *and* $\text{argmax}_{p\in P,x\in X}\Pi(p,x,t)$ *are increasing in* (p,t,X) *and in* (t,P,X)*, respectively.*

Proof. Because $k(x,t)$ is submodular, for part (a) it suffices by (3.3.6) to show that $(p-c)\mu(p,x,t)$ is supermodular in (p,x,t). Pick (p',x',t') and (p'',x'',t'') with $p''\geq c$ and $p'\geq c$. Without loss of generality, suppose that $p''\geq p'$ so $p'\vee p''=p''$ and $p'\wedge p''=p'$. Then

$$\begin{aligned}
&(p''-c)\mu(p'',x'\vee x'',t'\vee t'')+(p'-c)\mu(p',x'\wedge x'',t'\wedge t'')\\
&\quad -(p'-c)\mu(p',x',t')-(p''-c)\mu(p'',x'',t'')\\
&=(p''-c)(\mu(p'',x'\vee x'',t'\vee t'')+\mu(p',x'\wedge x'',t'\wedge t'')\\
&\quad -\mu(p',x',t')-\mu(p'',x'',t''))\\
&\quad +(p''-p')(\mu(p',x',t')-\mu(p',x'\wedge x'',t'\wedge t''))\\
&\geq 0,
\end{aligned}$$

where the equality is an identity and the inequality follows because $p''\geq c$, $\mu(p,x,t)$ is supermodular in (p,x,t), $p''\geq p'$, and $\mu(p,x,t)$ is increasing in

(x, t). This establishes part (a). Part (b) and part (c) follow from part (a) and from Theorem 2.7.6 and Theorem 2,8,2, respectively. □

The model of Example 3.3.5 specializes that of Example 3.3.3 by having the coefficient of the linear production cost be a constant parameter rather than a variable, and is broader by permitting price to be either a decision variable or a variable parameter rather than just a decision variable. A significant distinction is that price increases with the variable parameter in Theorem 3.3.3, while in Theorem 3.3.2 price decreases with the variable parameter. Theorem 3.3.3 assumes that the demand function is supermodular in (p, x, t), while Theorem 3.3.2 assumes that the demand function is log-supermodular in $(-p, x, t)$. Furthermore, the nature of the ordering comparing optima is somewhat stronger here than in part (e) of Theorem 3.3.2. As with Theorem 3.3.2, Theorem 3.3.3 does not assume that the demand function $\mu(p, x, t)$ is decreasing in p.

Example 3.3.6 is a special case of Example 3.3.5. This example extends a model of Topkis [1979], given in the context of a noncooperative game, by the inclusion of the decision variable q determining perceived product quality. See Subsection 4.4.1.

Example 3.3.6. A firm produces a single product, and other firms market substitute products. The firm sets the price p of its product, and ρ in R^m is the vector of prices of the substitute products marketed by other firms. Furthermore, the firm chooses the quality q that the market perceives its product to have. (This perceived quality may be real as resulting from capital investment and quality control, or it may only be a result of the level of advertising.) The firm's demand function is $\mu(p, q, \rho)$, which is increasing in q. Because the products are substitutes, $\mu(p, q, \rho)$ is increasing in ρ. The demand $\mu(p, q, \rho)$ is supermodular in (p, q), indicating that price and quality are substitute attributes in the market. That is, an improvement in quality has a greater impact on demand at higher prices, and a price reduction has a greater impact on demand for a lower quality product. Also, $\mu(p, q, \rho)$ has increasing differences in $((p, q), \rho)$, reflecting the properties that lower levels for the prices of competitors' substitute products increase the sensitivity of demand for the firm's product to its price and decrease the sensitivity of demand for the firm's product to its quality. The firm has a unit production cost c, where c is a constant parameter. There is a cost $k(q)$ for investments, procedures, and/or advertising that would lead the market to perceive a quality q for the firm's product. Part (c) of Theorem 3.3.3 applies directly with the decision variable $x = q$, the variable parameter $t = \rho$, and $k(x, t) = k(q)$, so the optimal price p and perceived quality q increase with the prices ρ of the substitute products.

The model presented in Example 3.3.7 and analyzed in Theorem 3.3.4 is a version of the Subsection 3.3.1 model that treats price as a variable (either a decision variable or a variable parameter), has a demand function that depends only on price, and has a general production cost function.

Example 3.3.7. A firm operates according to the general model of Subsection 3.3.1, with the following refinements. The market price p is a variable, either a decision variable or a variable parameter. The demand for the firm's product is $\mu(p)$, which depends on the variable p but not on the decision variable x nor on the variable parameter t. The firm's profit is

$$\Pi(p, x, t) = p\mu(p) - c(\mu(p), x, t) - k(x, t). \tag{3.3.7}$$

Theorem 3.3.4. *Consider the model of Example* 3.3.7 *with the profit function* $\Pi(p, x, t)$ *of* (3.3.7). *Suppose that* X *and* T *are sublattices,* $\mu(p)$ *is decreasing in* p*,* $c(z, x, t)$ *is submodular in* $(-z, x, t)$*, and* $k(x, t)$ *is submodular in* (x, t)*. Then,*
(a) $\Pi(p, x, t)$ *is supermodular in* (p, x, t)*;*
(b) $\max_{x\in X} \Pi(p, x, t)$ *and* $\max_{p\in P, x\in X} \Pi(p, x, t)$ *are supermodular in* (p, t) *and in* t*, respectively; and*
(c) $\operatorname{argmax}_{x\in X} \Pi(p, x, t)$ *and* $\operatorname{argmax}_{p\in P, x\in X} \Pi(p, x, t)$ *are increasing in* (p, t, X) *and in* (t, P, X)*, respectively.*

Proof. Because $c(z, x, t)$ is submodular in $(-z, x, t)$ and $\mu(p)$ is decreasing in p, $c(\mu(p), x, t)$ is submodular in (p, x, t). Then part (a) holds by (3.3.7) because $k(x, t)$ is submodular in (x, t). Part (b) and part (c) follow from part (a) and from Theorem 2.7.6 and Theorem 2.8.2, respectively. □

Theorem 3.3.4 is broader than Theorem 3.3.3, in the sense that the production cost is a general submodular function of the negative of the level of production and the non-price variables rather than being linear with a constant parameter coefficient. However, the demand function here can depend only on price, with which it must be decreasing, while in Theorem 3.3.3 the demand function could be a general supermodular function of price and the other variables.

For the case where the production cost is linear so $c(z, x, t) = c(x, t)z$, the hypotheses on $c(z, x, t)$ in Theorem 3.3.4 reduce to $c(x, t)$ being increasing and submodular in (x, t).

Example 3.3.8 is a special case of Example 3.3.7. This example is a version of Milgrom, Qian, and Roberts [1991], with price considered explicitly.

Example 3.3.8. A firm produces a single product in each of a sequence of periods. For each period, the firm myopically optimizes its profit function for that period. The demand $\mu(p)$ in each period depends only on the price p, a decision variable, and is decreasing in p. The other decision variable is θ, the level of technology used in the firm's production processes. The variable parameters in each period are the level of technology ϑ in the previous period and the current level of knowledge η publicly and privately available to the firm. The cost of producting an amount z is $c(z, \theta, \eta)$, which is submodular in (z, θ, η). This submodularity assumption means that the marginal cost of any additional production is decreasing with the levels of technology and knowledge and that the cost reduction resulting from a higher level of technology is greater with a greater amount of knowledge. The cost associated with having a production technology at level θ is $k(\theta, \eta, \vartheta)$, which has decreasing differences in $(\theta, (\eta, \vartheta))$. This decreasing differences assumption means that the marginal cost for using a higher level of technology decreases with the amount of knowledge and with the previous level of technology. The profit function is $p\mu(p) - c(\mu(p), \theta, \eta) - k(\theta, \eta, \vartheta)$, and part (c) of Theorem 3.3.4 with $x = -\theta$ and $t = (-\eta, -\vartheta)$ implies that the optimal price p and technology θ are decreasing and increasing, respectively, with the state of knowledge η, with the previous level of technology ϑ, and with the set of feasible technologies.

Now consider the multiperiod nature of the example and let the periods be indexed by τ. Suppose that the optimal solution in each period is unique. (This supposition can be relaxed if one is careful about which optimal solution is selected.) Denote the variable parameters η and ϑ and the optimal decision variables p and θ in period τ as $\eta(\tau)$, $\vartheta(\tau)$, $p(\tau)$, and $\theta(\tau)$, respectively. Then $\vartheta(\tau) = \theta(\tau - 1)$. The knowledge at the beginning of any period is increasing in the knowledge, technology, and production for the previous period, so $\eta(\tau) = h(\eta(\tau-1), \theta(\tau-1), p(\tau-1))$ where $h(\eta(\tau-1), \theta(\tau-1), p(\tau-1))$ is increasing in $(\eta(\tau-1), \theta(\tau-1), -p(\tau-1))$. The **momentum** property is that if there is any period τ' for which $\eta(\tau') \geq \eta(\tau' - 1)$ and $\vartheta(\tau') \geq \vartheta(\tau' - 1)$, then $(-p(\tau), \theta(\tau), \eta(\tau), \vartheta(\tau)) \geq (-p(\tau - 1), \theta(\tau - 1), \eta(\tau - 1), \vartheta(\tau - 1))$ for each $\tau \geq \tau'$. This follows by induction. Pick any $\tau \geq \tau'$ and suppose that $\eta(\tau) \geq \eta(\tau - 1)$ and $\vartheta(\tau) \geq \vartheta(\tau - 1)$. By the monotonicity of the decision variables with the variable parameters in each period, $(-p(\tau), \theta(\tau)) \geq (-p(\tau - 1), \theta(\tau - 1))$ and so $\eta(\tau + 1) = h(\eta(\tau), \theta(\tau), p(\tau)) \geq h(\eta(\tau - 1), \theta(\tau - 1), p(\tau - 1)) = \eta(\tau)$ and $\vartheta(\tau + 1) = \theta(\tau) \geq \theta(\tau - 1) = \vartheta(\tau)$. From each period to the next, the firm increases its sales and production by decreasing price, increases the level of production technology, and increases available knowledge.

The model presented in Example 3.3.9 and analyzed in Theorem 3.3.5 is a version of the Subsection 3.3.1 model that treats price as a variable (either a decision variable or a variable parameter), has a demand function that depends on price and on a limited component of the other variables, and has a production cost function depending on the level of production and on a limited component of the non-price variables.

Example 3.3.9. A firm operates according to the general model of Subsection 3.3.1, with the following refinements. The market price p is a variable, either a decision variable or a variable parameter. There is another variable y, which is restricted to be in a chain Y included in R^r and which may be either a decision variable or a variable parameter. The decision variable x does not include the variable y if y is a decision variable. The variable parameter t in T does not include the variable y if y is a variable parameter. The demand for the firm's product is $\mu(p, y)$, which depends on the variables p and y but not on the decision variable x nor on the variable parameter t. The cost of producing an amount z of the product is $c(z, y)$, which depends on the variable y. There are other costs, $k(y, x, t)$, depending on y, x, and t, but not on the level of production. The firm's profit is

$$\Pi(p, y, x, t) = p\mu(p, y) - c(\mu(p, y), y) - k(y, x, t). \tag{3.3.8}$$

Theorem 3.3.5. *Consider the model of Example* 3.3.9 *with the profit function* $\Pi(p, y, x, t)$ *of* (3.3.8). *Suppose that X and T are sublattices, Y is a chain, $\mu(p, y)$ is increasing in $(-p, y)$ and supermodular in (p, y), $pz - c(z, y)$ is increasing in z, $c(z, y)$ is convex in z and supermodular in (z, y), and $k(y, x, t)$ is submodular in (y, x, t). Then,*
(a) $\Pi(p, y, x, t)$ *is supermodular in* (p, y, x, t)*;*
(b) $\max_{y\in Y, x\in X} \Pi(p, y, x, t)$*,* $\max_{p\in P, x\in X} \Pi(p, y, x, t)$*, and* $\max_{p\in P, y\in Y, x\in X} \Pi(p, y, x, t)$ *are supermodular in (p, t), in (y, t), and in t, respectively; and*
(c) $\operatorname{argmax}_{y\in Y, x\in X} \Pi(p, y, x, t)$*,* $\operatorname{argmax}_{p\in P, x\in X} \Pi(p, y, x, t)$*, and* $\operatorname{argmax}_{p\in P, y\in Y, x\in X} \Pi(p, y, x, t)$ *are increasing in (p, t, Y, X), in (y, t, P, X), and in (t, P, Y, X), respectively.*

Proof. Because $k(y, x, t)$ is submodular, for part (a) it suffices by (3.3.8) to show that $p\mu(p, y) - c(\mu(p, y), y)$ is supermodular in (p, y). Pick p'', p', y'', and y' with $p'' > p'$ and $y'' > y'$. Then

$$\begin{aligned}
&p''\mu(p'', y'') - c(\mu(p'', y''), y'') + p'\mu(p', y') - c(\mu(p', y'), y') \\
&\qquad - (p''\mu(p'', y') - c(\mu(p'', y'), y')) - (p'\mu(p', y'') - c(\mu(p', y''), y'')) \\
&\quad = p''\mu(p'', y'') - c(\mu(p'', y''), y'') \\
&\qquad - (p''(\mu(p'', y') + \mu(p', y'') - \mu(p', y')) \\
&\qquad - \ c(\mu(p'', y') + \mu(p', y'') - \mu(p', y'), y''))
\end{aligned}$$

$$
\begin{aligned}
&+ (p'' - p')(\mu(p', y'') - \mu(p', y')) \\
&+ c(\mu(p', y''), y'') - c(\mu(p'', y') + \mu(p', y'') - \mu(p', y'), y'') \\
&- (c(\mu(p', y'), y'') - c(\mu(p'', y'), y'')) \\
&+ c(\mu(p', y'), y'') - c(\mu(p'', y'), y'') \\
&- (c(\mu(p', y'), y') - c(\mu(p'', y'), y')) \\
\geq\ & 0,
\end{aligned}
$$

where the equality is an identity and the second term is nonnegative since the first three lines of the second term are nonnegative because $\mu(p, y)$ is supermodular in (p, y) and $pz - c(z, y)$ is increasing in z, the fourth line is nonnegative because $\mu(p, y)$ is increasing in y, the fifth and sixth lines are nonnegative because $\mu(p, y)$ is increasing in $(-p, y)$ and $c(z, y)$ is convex in z, and the seventh and eighth lines are nonnegative because $\mu(p, y)$ is increasing in $-p$ and $c(z, y)$ is supermodular in (z, y). This establishes part (a). Part (b) and part (c) follow from part (a) and from Theorem 2.7.6 and Theorem 2.8.2, respectively. □

Theorem 3.3.5 is broader than Theorem 3.3.3, in the sense that the production cost is convex and can depend on a variable in addition to the level of production rather than being linear with a constant parameter coefficient. It is broader than Theorem 3.3.4, in the sense that the demand function can depend on a variable in addition to price. However, other generality is lost, since that variable in Theorem 3.3.5 is restricted to a chain.

For the case where the production cost is linear so $c(z, y) = c(y)z$, the hypotheses on $c(z, y)$ in Theorem 3.3.5 reduce to $p \geq c(y)$ for each y and $c(y)$ is increasing in y.

In the context of competitive models with substitute products, Milgrom and Shannon [1994] and Vives [1990] show that the optimal value of the decision variable p is increasing with the variable parameter y under conditions somewhat similar to those of Theorem 3.3.5. In those models, y represents the vector of prices of the substitute products, and the production cost depends only on the level of production (that is, $c(z, y) = c(z)$). Furthermore, the result of Milgrom and Shannon [1994] is based on ordinal methods, makes the weaker assumption that $\mu(p, y)$ is log-supermodular rather than supermodular in (p, y), and does without the assumption that $pz - c(z, y)$ is increasing in z.

Example 3.3.10 is a special case of Example 3.3.9.

Example 3.3.10. A firm produces a single product of quality q, which it sells for price p. The demand $\mu(p, q)$ is decreasing with price p and increasing with quality q. The demand $\mu(p, q)$ is supermodular in (p, q), indicating that price and quality are substitute attributes in the market; that is, quality improvement has a greater impact at higher prices and a price reduction has a greater impact given lower quality. The cost of producting an amount z with quality q is $c(z, q)$, which is supermodular in (z, q). This supermodularity means that the marginal cost of producing additional product is greater for product of higher quality. Furthermore, the operating profit from sales z, $pz - c(z, q)$, is increasing in z, and $c(z, q)$ is convex in z so the marginal cost of production is increasing for any given quality q. If the market would only accept product of a given quality q, so quality is a variable parameter, and if price is a decision variable, then by part (c) of Theorem 3.3.5 the firm's optimal price increases with that quality. This property contrasts with Example 3.3.2 where, under the rather different hypotheses of the modified Milgrom and Roberts [1990b] model, optimal price decreases with quality. As another version of this example, if price p is a variable parameter determined by the market or by a regulator and if quality is a decision variable, then by part (c) of Theorem 3.3.5 optimal quality for the firm's product increases with that price. As a third version, suppose that both price and quality are decision variables, that time τ is a variable parameter, and that there is a capital cost $k(q, \tau)$ associated with being able to produce product of quality q at time τ. Suppose that $k(q, \tau)$ is submodular in (q, τ), so the marginal capital needed to produce a higher quality product decreases as time goes on. By part (c) of Theorem 3.3.5, price and quality both increase with time.

3.3.3 Necessary Conditions

With the particular exception of the log-supermodularity assumption of Theorem 3.3.2 (which is addressed with Theorem 3.3.7), Theorem 3.3.6 shows that hypotheses of Subsection 3.3.2 are necessary for the firm's decision problems to have increasing optimal solutions. (In relating the conditions of Theorem 3.3.6 with those of Subsection 3.3.2, recall the convention of Subsection 3.3.2 that all statements of hypotheses yielding increasing optimal solutions based on Theorem 2.8.2 are required to hold only for each subset of variable parameters that is a chain.) Part (a), part (b), part (c), part (d), and part (e) of Theorem 3.3.6 establish the necessity of hypotheses made throughout Subsection 3.3.2 that the cost function $k(x, t)$ is submodular in x and has decreasing differences in (x, t); that the demand function $\mu(p, x, t)$ is supermodular in x, has increasing differences in (x, t), and is increasing in (x, t); and that the production cost function $c(z, x, t)$ is submodular in x and has decreasing differences in (x, t). Part (c) and part (d) of Theorem 3.3.6 establish the necessity of $\mu(p, x, t)$ having increasing differences in (p, x) and in (p, t) for the third and fifth models of Subsection 3.3.2. Part (f) of Theorem 3.3.6 establishes the necessity of the concavity of $c(z, x, t)$ in z for the first model of Subsection 3.3.2.

Theorem 3.3.6. *Consider the model summarized in the profit function of* (3.3.1).
(a) *Suppose that* $c(z, x, t) = cz$ *for a constant parameter* c *with* $c \neq p$ *and* $\mu(p, x, t) = \mu'(p, t) + q \cdot x$ *where* q *in* R^n *is a constant parameter, so*

$$\Pi(p, x, t) = (p - c)\mu'(p, t) + (p - c)q \cdot x - k(x, t).$$

If $\operatorname{argmax}_{x \in X} \Pi(p, x, t)$ *is increasing in* (t, X) *for each* q *in* R^n, *then* $k(x, t)$ *is submodular in* x *and has decreasing differences in* (x, t).
(b) *Suppose that* $c(z, x, t) = cz$ *for a constant parameter* c *with* $c < p$ *and* $k(x, t) = k'(t) + q \cdot x$ *where* q *in* R^n *is a constant parameter, so*

$$\Pi(p, x, t) = (p - c)\mu(p, x, t) - k'(t) - q \cdot x.$$

If $\operatorname{argmax}_{x \in X} \Pi(p, x, t)$ *is increasing in* (t, X) *for each* q *in* R^n, *then* $\mu(p, x, t)$ *is supermodular in* x *and has increasing differences in* (x, t).
(c) *Suppose that* $c(z, x, t) = cz$ *for a constant parameter* c *with* $c \leq p$ *and* $k(x, t) = k'(t) + q \cdot x$ *where* q *in* R^n *is a constant parameter, so*

$$\Pi(p, x, t) = (p - c)\mu(p, x, t) - k'(t) - q \cdot x.$$

If $\operatorname{argmax}_{x \in X} \Pi(p, x, t)$ *is increasing in* p *for each* X, *for each* q *in* R^n, *and for each* c *with* $c \leq p$, *then* $\mu(p, x, t)$ *is increasing in* x *and has increasing differences in* (p, x).
(d) *Suppose that* $c(z, x, t) = cz$ *for a constant parameter* c *with* $c \leq p$ *and* $\mu(p, x, t) = \mu'(p, x, t) + q$ *where* q *in* R^1 *is a constant parameter, so*

$$\Pi(p, x, t) = (p - c)\mu'(p, x, t) + qp - qc - k(x, t).$$

If $\operatorname{argmax}_{p \in P} \Pi(p, x, t)$ *is increasing in* t *for each* P, *for each* q *in* R^1, *and for each* c *with* $c \leq p$, *then* $\mu'(p, x, t)$ *is increasing in* t *and has increasing differences in* (p, t).
(e) *Suppose that* $\mu(p, x, t) = u$ *where* u *in* R^1 *is a constant parameter and* $k(x, t) = k'(t) + q \cdot x$ *where* q *in* R^n *is a constant parameter, so*

$$\Pi(p, x, t) = pu - c(u, x, t) - k'(t) - q \cdot x.$$

If $\operatorname{argmax}_{x \in X} \Pi(p, x, t)$ *is increasing in* (t, X) *for each* q *in* R^n, *then* $c(u, x, t)$ *is submodular in* x *and has decreasing differences in* (x, t).

(f) *Suppose that* $\mu(p, x, t)$ *is linear and strictly increasing in* (x, t), $n \geq 1$, $m \geq 1$, $c(z, x, t) = c(z)$ *where* $c(z)$ *is continuous, and* $k(x, t) = k'(t) + q \cdot x$ *where* q *in* R^n *is a constant parameter, so*

$$\Pi(p, x, t) = p\mu(p, x, t) - c(\mu(p, x, t)) - k'(t) - q \cdot x.$$

If $\text{argmax}_{x \in X} \Pi(p, x, t)$ *is increasing in* t *for each nonempty sublattice* X *and for each* q *in* R^n, *then* $c(z)$ *is concave in* z.

Proof. Part (a), part (b), and part (e) follow directly from Theorem 2.8.10 and Theorem 2.8.11.

Under the hypotheses of part (c), Theorem 2.8.11 implies that $(p - c)\mu(p, x, t)$ has increasing differences in (p, x). Pick p'', p', x'', and x' with $p'' > p'$ and $x'' > x'$. Because $(p - c)\mu(p, x, t)$ has increasing differences in (p, x),

$$\begin{aligned}
0 &\leq (p'' - c)(\mu(p'', x'', t) - \mu(p'', x', t)) \\
&\quad - (p' - c)(\mu(p', x'', t) - \mu(p', x', t)) \\
&= (p' - c)((\mu(p'', x'', t) - \mu(p'', x', t)) \\
&\quad - (\mu(p', x'', t) - \mu(p', x', t))) \\
&\quad + (p'' - p')(\mu(p'', x'', t) - \mu(p'', x', t)).
\end{aligned} \tag{3.3.9}$$

Letting $c = p'$, (3.3.9) implies that $\mu(p, x, t)$ is increasing in x. Letting c be sufficiently small, (3.3.9) implies that $\mu(p, x, t)$ has increasing differences in (p, x).

Under the hypotheses of part (d), Theorem 2.8.11 implies that $(p - c)\mu'(p, x, t)$ has increasing differences in (p, t). It now follows as in the proof of part (c) that $\mu'(p, x, t)$ is increasing in t and has increasing differences in (p, t).

Under the hypotheses of part (f), Theorem 2.8.11 implies that $c(\mu(p, x, t))$ has decreasing differences in (x, t). Because $c(\mu(p, x, t))$ has decreasing differences in (x, t), $c(z)$ is continuous, $\mu(p, x, t)$ is linear and strictly increasing in (x, t), $n \geq 1$, and $m \geq 1$, $c(z)$ is concave in z by part (c) of Lemma 2.6.2. □

Theorem 3.3.7 shows that if T is a chain then the log-supermodularity of the demand function in $(-p, t)$, as assumed in Theorem 3.3.2 and in Corollary 3.3.2, is necessary for the firm's decision problem in Example 3.3.3 and in Example 3.3.4 to have optimal solutions where price decreases with the parameter t.

Theorem 3.3.7. *If T is a chain, P is a subset of R^1, $\mu(p, t)$ is positive and decreasing in p on P for each t in T, and* $\text{argmax}_{p\in P'}(p-c)\mu(p, t)$ *is decreasing in t on T for each real c and for each subset P' of P, then $\mu(p, t)$ is log-supermodular in $(-p, t)$ on $(-P)\times T$ (where $-P = \{-p : p \in P\}$).*

Proof. Suppose that $\mu(p, t)$ is not log-supermodular in $(-p, t)$. Then there exist p', p'', t', and t'' such that $p'' > p'$, $t'' > t'$, and

$$\log \mu(p'', t') + \log \mu(p', t'') < \log \mu(p'', t'') + \log \mu(p', t').$$

Therefore,

$$\mu(p'', t')\mu(p', t'') < \mu(p'', t'')\mu(p', t'),$$

and so, using the property that $\mu(p, t)$ is decreasing in p,

$$1 \le \mu(p', t'')/\mu(p'', t'') < \mu(p', t')/\mu(p'', t'). \tag{3.3.10}$$

By (3.3.10), one can pick $c' < p'$ such that

$$\mu(p', t'')/\mu(p'', t'') < (p'' - c')/(p' - c') < \mu(p', t')/\mu(p'', t'). \tag{3.3.11}$$

By (3.3.11),

$$(p' - c')\mu(p', t'') < (p'' - c')\mu(p'', t'') \tag{3.3.12}$$

and

$$(p'' - c')\mu(p'', t') < (p' - c')\mu(p', t'). \tag{3.3.13}$$

Let $P' = \{p', p''\}$. By (3.3.12),

$$\text{argmax}_{p\in P'}(p - c')\mu(p, t'') = \{p''\}. \tag{3.3.14}$$

By (3.3.13),

$$\text{argmax}_{p\in P'}(p - c')\mu(p, t') = \{p'\}. \tag{3.3.15}$$

Because $p'' > p'$ and $t'' > t'$, (3.3.14) and (3.3.15) show that $\text{argmax}_{p\in P'}(p - c')\mu(p, t)$ is not decreasing in t. □

3.4 Transportation and Transshipment Problems

Subsection 3.4.1 uses general properties for maximizing a supermodular function on a lattice to examine the structure of the transportation problem, with the parameters taken to be the capacities at the factories and the demands of the customers (or the set of operating factories and the set of customers in the market). Complementarity and substitutability properties hold, and the optimal dual variables are monotone with respect to problem parameters. Subsection 3.4.2 gives examples showing that seemingly natural extensions of these complementarity and substitutability properties fail to hold in general for the two-stage transshipment problem.

3.4.1 Transportation Problem

Consider the **transportation problem**, where a firm produces a single product at each of a finite set of factories I to satisfy the demand of each of a finite set of customers K. The maximum production capacity at factory i is $S_i \geq 0$. The demand of customer k is $D_k \geq 0$. The product can be shipped from any factory to any customer. There is a unit transportation cost $c_{i,k}$ for shipping the product from factory i to customer k. The objective of the firm is to satisfy all customer demand from production at its factories in such a way as to minimize total shipping costs. Assume that $\sum_{i \in I} S_i \geq \sum_{k \in K} D_k$, so that the problem of the firm is feasible and an optimal solution exists. Let $x_{i,k}$ be the amount shipped from factory i to customer k. The problem of the firm is to

$$\text{minimize } \sum_{i \in I} \sum_{k \in K} c_{i,k} x_{i,k}$$

subject to

$$-\sum_{k \in K} x_{i,k} \geq -S_i \qquad \text{for each } i \text{ in } I,$$

$$\sum_{i \in I} x_{i,k} \geq D_k \qquad \text{for each } k \text{ in } K,$$

and

$$x_{i,k} \geq 0 \qquad \text{for all } i \text{ in } I \text{ and } k \text{ in } K.$$

For the (primal) transportation problem stated above, the **dual** (Dantzig [1963]) is to

$$\text{maximize } \sum_{i \in I} (-S_i) w_i + \sum_{k \in K} D_k v_k$$

subject to

$$v_k - w_i \leq c_{i,k} \quad \text{for all } i \text{ in } I \text{ and } k \text{ in } K,$$

$$w_i \geq 0 \quad \text{for each } i \text{ in } I,$$

and

$$v_k \geq 0 \quad \text{for each } k \text{ in } K.$$

Two related sets of parameters are of interest here. The first set of parameters consists of the collection of all capacities S_i and demands D_k, where the sets of factories I and customers K are constant. Fix the set of factories I and the set of customers K, and let $S = \{S_i : i \in I\}$, $D = \{D_k : k \in K\}$, $w = \{w_i : i \in I\}$, and $v = \{v_k : k \in K\}$. For any vectors of supplies S and demands D, let $g(S, D)$ be the optimal value of the objective function in the transportation problem and let $Y(S, D)$ be the collection of all optimal solutions (w, v) for the dual of the transportation problem. The second set of parameters consists of the set I of open factories and the set K of customers in the market, where the capacities and demands for potentially open factories and potentially available customers are constant. For this second case, suppose that there are a set I' of factories that may potentially be operating and a set K' of customers who may potentially be in the market, with $S' = \{S_i : i \in I'\}$ and $D' = \{D_k : k \in K'\}$. The sets I' and K' are treated as constants, and the set I of operating factories and the set K of customers in the market are subsets of I' and K', respectively. For any subsets I and K of I' and K', respectively, let $h(I, K)$ be the optimal value of the objective function in the transportation problem with a set of factories I and a set of customers K and let $Z(I, K)$ be the collection of all optimal solutions (w, v) for the dual of the transportation problem. The first set of parameters (S and D) is more general because adding a new factory i corresponds to increasing its capacity from 0 to its operating capacity S_i and adding a new customer k corresponds to increasing that customer's demand from 0 to its actual demand D_k. For this reason, the proof for Theorem 3.4.1 below only gives the proof for the first set of parameters (S and D). Nevertheless, results involving the sets of factories I and customers K as parameters are of direct interest, so results for these are also stated explicitly in Theorem 3.4.1.

The results of Theorem 3.4.1 have interpretations in terms of complements and substitutes. Part (a) of Theorem 3.4.1 establishes the supermodularity of the optimal cost function $g(S, D)$ in $(-S, D)$ for the transportation problem. Theorem 2.6.1 then implies that this optimal cost function has increasing differences as a function of the capacities of any two factories, increasing differences as a function of the demands of any two customers, and decreasing differences as a function of the capacity of any factory and the demand of any customer. Therefore, the capacities of any two factories are substitutes, the demands of any two customers are substitutes, and the capacity of any factory and the demand of any customer are complements. That is, the cost decrease due to greater capacity at any factory decreases with the capacities of the other factories and increases with the demands of the customers, and the cost increase due to greater demand from any customer decreases with the capacities of the factories and increases with the demands of the other customers. It likewise follows from the supermodularity of the optimal cost $h(I, K)$ in $(I' \setminus I, K)$, as in part (a) of Theorem 3.4.1, that any two factories are substitutes, any two customers are substitutes, and any factory and any customer are complements. That is, the cost decrease due to opening a new factory decreases with the set of other open factories and increases with the set of customers in the market, and the cost increase due to an additional customer decreases with the set of open factories and increases with the set of other customers in the market. Part (b) and part (c) of Theorem 3.4.1 establish the monotonicity of the optimal dual variables with respect to the parameters in the transportation problem. Because dual variables represent the marginal change in the optimal primal objective function value with respect to changes in the elements on the right-hand side of the primal inequality constraints, interpretations of part (b) and part (c) of Theorem 3.4.1 in terms of substitutability and complementarity are similar to the preceding interpretation of part (a).

Samuelson [1952] asserts a version of part (b) of Theorem 3.4.1 with $-S$ as the parameter. Shapley [1962] considers complements and substitutes in the **assignment problem**, which is a transportation problem with each factory's supply and each customer's demand equal to 1. Shapley [1962] shows that the optimal cost function has increasing differences with respect to any two factories (so any two factories are substitutes), increasing differences with respect to any two customers (so any two customers are substitutes), and decreasing differences with respect to any factory and any customer (so any factory and any customer are complements). Shapley [1962] further comments that these results for the assignment problem could be extended to the transportation problem. Erlenkotter [1969] establishes that if $(-S', D') \leq (-S'', D'')$ and (w', v') is in $Y(S', D')$, then there exists (w'', v'') in $Y(S'', D'')$ such that $(w', v') \leq (w'', v'')$. Part (b) of Theorem 3.4.1 phrases a monotonicity result for the optimal dual solutions more generally in terms of the induced set ordering $\sqsubseteq$ and provides a more succinct proof based on the underlying structure of the problem. Erlenkotter [1969] uses the monotonicity of the optimal dual solutions to extend the results of Shapley [1962] from the assignment problem to the transportation problem. By Theorem 2.6.1 and Corollary 2.6.1, this extension is equivalent to part (a) of Theorem 3.4.1. Akinc and Khumawala [1977] use the property that factories are substitutes in the transportation problem as a basis for an efficient algorithm for locating factories in a transportation network. Gale and Politof [1981] give a version of part (a) of Theorem 3.4.1 for more general networks.

Theorem 3.4.1. *Consider the transportation problem with* $S \geq 0$*,* $D \geq 0$*, and* $\sum_{i \in I} S_i \geq \sum_{k \in K} D_k$*. Then*
(a) *the optimal cost* $g(S, D)$ $(h(I, K))$ *is supermodular in* $(-S, D)$ $((I' \setminus I, K))$*;*
(b) *the set of optimal dual solutions* $Y(S, D)$ $(Z(I, K))$ *is increasing in* $(-S, D)$ $((I' \setminus I, K))$*; and*
(c) *the set of optimal dual solutions* $Y(S, D)$ $(Z(I, K))$ *has a least element, and the least element of* $Y(D, S)$ $(Z(I, K))$ *is increasing in* $(-S, D)$ $((I' \setminus I, K))$*.*

Proof. The transportation problem has a feasible solution because $\sum_{i \in I} S_i \geq \sum_{k \in K} D_k$, and the feasible region for the transportation problem is bounded because $\sum_{k \in K} x_{i,k} \leq S_i$ for each i in I and $x_{i,k} \geq 0$ for all i in I and k in K. Therefore, some optimal solution exists for the dual of the transportation problem and $Y(S, D)$ is nonempty (Dantzig [1963]). The objective function for the dual problem is supermodular in $(w, v, -S, D)$ by part (d) of Example 2.6.2 and part (b) of Lemma 2.6.1. The set of feasible variables (w, v) for the dual problem is a sublattice of $R^{|I|+|K|}$ as in part (b) of Example 2.2.7 and does not depend on $(-S, D)$. Part (a) follows from applying Theorem 2.7.6 to the dual problem. Part (b) follows from Theorem 2.8.2. By Theorem 2.7.1, $Y(S, D)$ is a sublattice of $R^{|I|+|K|}$. Because the dual variables are constrained to be nonnegative, $Y(S, D)$ is bounded below. By Corollary 2.3.1, $Y(S, D)$ has a least element. Therefore, part (c) follows from part (b) and Lemma 2.4.2. □

Establishing the results of Theorem 3.4.1 relies on two transformations. One is the multiplication by -1 and the reversal of the inequality of the constraints ($\sum_{k \in K} x_{i,k} \leq S_i$ for each i in I) having to do with limited supplies at factories before expressing these inequalities as the first set of inequality constraints ($-\sum_{k \in K} x_{i,k} \geq -S_i$ for each i in I) in the formulation of the transportation problem. The other is the transformation from the primal transportation problem to its dual. The general results of Chapter 2 for maximizing a supermodular function on a lattice do not apply to any version of the primal problem, even though the result of part (a) of Theorem 3.4.1 is a property of the primal problem. Furthermore, the application in Theorem 3.4.1 of the results of Chapter 2 to the dual problem depends on the form of the dual, which in turn depends on the form of the primal. The transformation multiplying the inequalities ($\sum_{k \in K} x_{i,k} \leq S_i$ for each i in I) by -1 (and reversing the inequality) in the primal problem has the effect of multiplying the corresponding dual variables and their coefficients by -1, and the set of feasible solutions for the dual would not be a sublattice without this.

The **generalized transportation problem** (Ahuja, Magnanti, and Orlin [1993]) is similar to the transportation problem except that for each factory i and customer k there is a positive constant $\gamma_{i,k}$ such that if $x_{i,k}$ units of the product are shipped from factory i to customer k then $\gamma_{i,k}x_{i,k}$ units of the product arrive at customer k. A version of Theorem 3.4.1 holds for the generalized transportation problem, with an analysis very similar to that given for the transportation problem. (The dual of the generalized transportation problem is a sublattice of $R^{|I|+|K|}$ by part (b) of Example 2.2.7.)

3.4.2 Transshipment Problem

Now consider a single product **two-stage transshipment problem** with a finite set of factories I, a finite set of warehouses J, and a finite set of customers K. Factory i has capacity $S_i \geq 0$. Customer k has demand $D_k \geq 0$. The product can be shipped from any factory to any warehouse and from any warehouse to any customer, but not between any other ordered pair of locations. The total amount of the product shipped into each warehouse must equal the total amount of the product shipped out of that warehouse. The unit cost of shipping the product from factory i to warehouse j is $a_{i,j}$. The unit cost of shipping the product from warehouse j to customer k is $b_{j,k}$. The objective of the firm is to satisfy all customer demand from production at its factories so that total shipping costs are minimized.

Let $y_{i,j}$ be the amount shipped from factory i to warehouse j, and let $z_{j,k}$ be the amount shipped from warehouse j to customer k.

The problem of the firm is to

$$\text{minimize } \textstyle\sum_{i\in I}\sum_{j\in J} a_{i,j}y_{i,j} + \sum_{j\in J}\sum_{k\in K} b_{j,k}z_{j,k}$$

subject to

$$-\textstyle\sum_{j\in J} y_{i,j} \geq -S_i \quad \text{for each } i \text{ in } I,$$

$$\textstyle\sum_{i\in I} y_{i,j} - \sum_{k\in K} z_{j,k} = 0 \quad \text{for each } j \text{ in } J,$$

$$\textstyle\sum_{j\in J} z_{j,k} \geq D_k \quad \text{for each } k \text{ in } K,$$

$$y_{i,j} \geq 0 \quad \text{for all } i \text{ in } I \text{ and } j \text{ in } J,$$

and

$$z_{j,k} \geq 0 \quad \text{for all } j \text{ in } J \text{ and } k \text{ in } K.$$

If the collection of warehouses J is held fixed, then the two-stage transshipment problem is equivalent to a transportation problem with a set of factories I and a set of customers K and a unit cost $\min_{j\in J}(a_{i,j} + b_{j,k})$ for shipping from factory i to customer k. This is so because in an optimal solution for the two-stage transshipment problem any product produced at factory i that eventually satisfies some of the demand of customer k travels from factory i to some warehouse j' and then from warehouse j' to customer k, where j' minimizes $a_{i,j} + b_{j,k}$ over all j in J. Therefore, relationships among pairs of elements taken from the set of all factories and all customers are the same in the two-stage transshipment problem as the relationships indicated by part (a) of Theorem 3.4.1 for the transportation problem. That is, any two factories are substitutes, any two customers are substitutes, and any factory and any customer are complements.

Stated more abstractly, a conclusion of Theorem 3.4.1 is that for the transportation problem any two elements of the same type are substitutes and any two elements of different types are complements. This would likewise seem to be a plausible property for the two-stage transshipment problem, and, indeed, the preceding paragraph notes that this generalization is valid for pairs of elements not including any warehouse. However, as in Topkis [1984b], Example 3.4.1, Example 3.4.2, and Example 3.4.3 show, respectively, that for the two-stage transshipment problem two warehouses need not be substitutes, a warehouse and a customer need not be complements, and a factory and a warehouse need not be complements. The generalization of complementarity and substitutability properties for the transportation problem to the two-stage transshipment problem does not follow from a similar proof based on general properties of a supermodular function on a sublattice because if the two-stage transshipment problem is reformulated to include an upper bound on the flow of the product through each warehouse then the dual feasible set is not a sublattice. While the precise technical conditions of Theorem 2.7.6 and Theorem 2.8.2 are not necessary for all manifestations of complementarity, recognizing when any of these conditions fails to hold may lead to identifying unexpected anomalies as for the two-stage transshipment problem. Nagelhout and Thompson [1981] use the supposition of the substitutability of warehouses in the two-stage transshipment problem as a basis for concluding the validity of an algorithm for locating warehouses, similar to the algorithm of Akinc and Khumawala [1977] for locating factories in a transportation network. Example 3.4.1 shows that this supposition of the substitutability of warehouses is not correct, and an example of Topkis [1984b] shows that the algorithm of Nagelhout and Thompson [1981] need not locate warehouses optimally as claimed.

Example 3.4.1. Consider the two-stage transshipment problem with $I = \{i_1, i_2, i_3\}$, $J = \{j_1, j_2, j_3, j_4\}$, $K = \{k_1, k_2, k_3\}$, $S_{i_1} = S_{i_3} = D_{k_1} = D_{k_2} = D_{k_3} = 1$, $S_{i_2} = 2$, $a_{i_1,j_1} = a_{i_2,j_2} = a_{i_2,j_3} = a_{i_3,j_4} = b_{j_2,k_1} = b_{j_3,k_2} =$

$b_{j_3,k_3} = b_{j_4,k_3} = 0$, $b_{j_1,k_1} = 1$, and all other $a_{i,j}$ and $b_{j,k}$ taking on arbitrarily large values. Suppose that all factories, warehouses, and customers are present and held fixed with the possible exception of warehouses j_2 and j_4, and let T be the subset of $\{j_2, j_4\}$ that is present. Let $F(T)$ be the minimum cost in the two-stage transshipment problem viewed as a function of T. Then, $F(\emptyset) = 1$, $F(\{j_2\}) = 1$, $F(\{j_4\}) = 1$, and $F(\{j_2, j_4\}) = 0$. Therefore, $F(\{j_2\}) + F(\{j_4\}) = 2 > 1 = F(\{j_2, j_4\}) + F(\emptyset)$, and so warehouses j_2 and j_4 are not substitutes.

Example 3.4.2. Consider the two-stage transshipment problem with $I = \{i_1, i_2\}$, $J = \{j_1, j_2, j_3\}$, $K = \{k_1, k_2, k_3\}$, $S_{i_1} = D_{k_1} = D_{k_2} = D_{k_3} = 1$, $S_{i_2} = 2$, $a_{i_1,j_1} = a_{i_2,j_2} = a_{i_2,j_3} = b_{j_2,k_1} = b_{j_3,k_2} = b_{j_3,k_3} = 0$, $b_{j_1,k_1} = 1$, and all other $a_{i,j}$ and $b_{j,k}$ taking on arbitrarily large values. Suppose that all factories, warehouses, and customers are present and held fixed with the possible exception of warehouse j_2 and customer k_3, and let T be the subset of $\{j_2, k_3\}$ that is present. Let F(T) be the minimum cost in the two-stage transshipment problem viewed as a function of T. Then, $F(\emptyset) = 1$, $F(\{j_2\}) = 0$, $F(\{k_3\}) = 1$, and $F(\{j_2, k_3\}) = 1$. Therefore, $F(\{j_2\}) + F(\{k_3\}) = 1 < 2 = F(\{j_2, k_3\}) + F(\emptyset)$, and so warehouse j_2 and customer k_3 are not complements.

Example 3.4.3. Consider the two-stage transshipment problem with $I = \{i_1, i_2, i_3\}$, $J = \{j_1, j_2, j_3\}$, $K = \{k_1\}$, $S_{i_1} = S_{i_2} = S_{i_3} = D_{k_1} = 1$, $a_{i_1,j_1} = a_{i_2,j_2} = a_{i_3,j_3} = b_{j_2,k_1} = b_{j_3,k_1} = 0$, $b_{j_1,k_1} = 1$, and all other $a_{i,j}$ taking on arbitrarily large values. Suppose that all factories, warehouses, and customers are present and held fixed with the possible exception of factory i_3 and warehouse j_2, and let T be the subset of $\{i_3, j_2\}$ that is present. Let $F(T)$ be the minimum cost in the two-stage transshipment problem viewed as a function of T. Then, $F(\emptyset) = 1$, $F(\{i_3\}) = 0$, $F(\{j_2\}) = 0$, and $F(\{i_3, j_2\}) = 0$. Therefore, $F(\{i_3\}) + F(\{j_2\}) = 0 < 1 = F(\{i_3, j_2\}) + F(\emptyset)$, and so factory i_3 and warehouse j_2 are not complements.

3.5 Dynamic Economic Lot Size Production Models, Acyclic Networks

This section consists of two related subsections. Subsection 3.5.1 considers submodularity and increasing optimal solutions for the problem of finding minimum cost paths from one node to all other nodes in an acyclic network. Special use is made of the dynamic programming formulation for this problem. Subsection 3.5.2 applies the results of Subsection 3.5.1 to dynamic economic lot size production models (Eppen, Gould, and Pashigian [1969]; Wagner and

Whitin [1958]; Zangwill [1969]), clarifying the role of submodularity in monotonicity and planning horizon results for this class of problems.

3.5.1 Acyclic Networks

A (directed) **network** consists of a finite collection of **nodes** and a set of directed **edges**, where each edge joins an ordered pair of nodes and at most one edge joins any particular ordered pair of nodes. An edge from node i' to node i'' is denoted (i', i''). A (directed) **path** from node i' to node i'' consists of a sequence of nodes $\{i_1, i_2, \ldots, i_{k'}\}$ such that (i_k, i_{k+1}) is an edge for $k = 1, \ldots, k'-1$, $i_1 = i'$, and $i_{k'} = i''$. If there is a cost $c(i', i'')$ associated with each edge (i', i''), then the cost associated with any path is the sum of all costs associated with all edges on the path. A network is **acyclic** if the nodes can be represented $\{1, \ldots, n\}$ and there is an edge from node i' to node i'' if and only if $i' < i''$; that is, the set of edges is $\{(i', i'') : 1 \leq i' < i'' \leq n, i' \text{ and } i'' \text{ integer}\}$. As in part (d) of Example 2.2.7, this set of edges is a sublattice of R^2. (The present definition of an acyclic network is a simplification, for convenience, of a more general definition that would permit the set of edges to be any subset of $\{(i', i'') : 1 \leq i' < i'' \leq n, i' \text{ and } i'' \text{ integer}\}$. Using that more general definition would require the additional assumption for Theorem 3.5.1 below that the set of edges is a sublattice, as this property would no longer be implied by part (d) of Example 2.2.7.)

Consider an acyclic network with nodes $\{1, \ldots, n\}$, edges $\{(i', i'') : 1 \leq i' < i'' \leq n, i' \text{ and } i'' \text{ integer}\}$, and a cost $c(i', i'')$ associated with each edge (i', i''). One is interested in finding minimum cost paths and in determining properties of such paths from node 1 to each other node i. For $i = 1, \ldots, n$, let $f(i)$ be the cost of a minimum cost path from node 1 to node i. The values $f(i)$ may be found from the well-known dynamic programming recursion

$$f(i) = \min_{\{i:1 \leq i' \leq i-1\}}(f(i') + c(i', i)) \text{ for each } i > 1, \quad f(1) = 0. \tag{3.5.1}$$

One begins with $f(1) = 0$ and, given $f(1), \ldots, f(i-1)$ for any i with $1 < i \leq n$, the value of $f(i)$ is computed directly from (3.5.1). This process continues until $f(n)$ is computed. For $i = 2, \ldots, n$, let $J(i)$ be the set of all i' that are optimal given i in (3.5.1). It follows inductively that $f(i)$ as computed from (3.5.1) is indeed the cost of a minimum cost path from node 1 to node i, and that the minimum cost paths from node 1 to node i may be constructed by taking any node i' in $J(i)$, proceeding from node 1 to node i' on a minimum cost path, and then adding the edge (i', i) to that path. More precisely, a minimum cost path from node 1 to node i can be constructed tracing backwards by letting the last node on the path be i; letting the node before i on the path be any element $j(i)$ of $J(i)$; letting the node before $j(i)$ on the path be any element

$j(j(i))$ of $J(j(i))$; and continuing in this manner until node 1 is included in the path.

Theorem 3.5.1 develops structural properties when the cost function $c(i', i'')$ is submodular in (i', i''). Then, the penultimate node on a minimum cost path from node 1 to node i is increasing in i, and this property may be used to reduce the computation required in (3.5.1). Furthermore, if a minimum cost path from node 1 to some node i passes through node $i-1$, then for each $i' > i$ there exists a minimum cost path from node 1 to node i' that passes through node $i-1$. This last property is a **planning horizon** result (in a sense that is perhaps more meaningful in a temporal context as in Subsection 3.5.2), since observing a node i with $i-1$ in $J(i)$ yields the initial part (from node 1 through node $i-1$) of a minimum cost path from node 1 to any node i' with $i' > i$ without considering the particular values of $c(i'', i''')$ for any i'' and i''' with $i < i'''$. The application of general properties of submodular functions in Theorem 3.5.1 is facilitated by the transformation of the minimum cost path problem into the dynamic programming formulation of (3.5.1).

Theorem 3.5.1. *Suppose that* $c(i', i'')$ *is submodular in* (i', i'') *on the sublattice* $\{(i', i'') : 1 \le i' < i'' \le n,\ i' \text{ and } i'' \text{ integer}\}$ *of* R^2.
(a) *The set* $J(i)$ *of* i' *optimal given* i *in* (3.5.1) *is increasing in* i *for* i *in* $\{2, \ldots, n\}$.
(b) *If* $j(i-1)$ *is any element of* $J(i-1)$ *for* i *in* $\{3, \ldots, n\}$, *then the recursion*

$$f(i) = \min_{\{i' : j(i-1) \le i' \le i-1\}}(f(i') + c(i', i)) \text{ for each } i > 1, \quad f(1) = 0$$

can replace (3.5.1).
(c) *If* $j(i)$ *is the greatest (least) element of* $J(i)$, *then* $j(i)$ *is increasing in* i *for* i *in* $\{2, \ldots, n\}$.
(d) *If* $i-1$ *is in* $J(i)$ *for some node* i, *then for each integer* i' *with* $i \le i' \le n$ *there exists a minimum cost path from node* 1 *to node* i' *that passes through node* $i-1$.

Proof. Part (a) follows from Theorem 2.8.2.

If $j(i-1)$ is in $J(i-1)$, then for any i' in $J(i)$ the node $j(i-1) \vee i'$ is in $J(i)$ by part (a). Consequently, $\{i' : j(i-1) \le i' \le i-1\} \cap J(i)$ is nonempty and part (b) follows because one may optimize over $\{i' : j(i-1) \le i' \le i-1\}$ instead of over $\{i' : 1 \le i' \le i-1\}$ in (3.5.1).

Part (c) follows from part (a) of Theorem 2.8.3.

Let $j(i)$ be the greatest element of $J(i)$ for each node i. Suppose that $i-1$ is in $J(i)$ for some node $i > 1$, so $j(i) = i-1$. Pick any node i' with $i \le i' \le n$. Construct a minimum cost path from node 1 to node i' as follows. The last node on the path is i'. Beginning with the last node i', construct the nodes on the path in backwards order so that if node i'' is on the path for $i'' > 1$ then the node preceding i'' on the path is $j(i'')$. Continue this construction until node 1 is included in the path. This path is a minimum cost path from node 1 to node i' by (3.5.1) and the definition of $J(i)$. Let i''' be the least element of $\{i, i+1, \ldots, i'\}$ on the path. Then $j(i''')$ is on the path and $j(i''') \leq i-1$. However, $i-1 = j(i) \leq j(i''')$ by part (c). Therefore, $j(i''') = i-1$ and node $i-1$ is on this minimum cost path from node 1 to node i'. □

3.5.2 Dynamic Economic Lot Size Production Models

A firm produces a single product in each of n periods denoted $i = 1, \ldots, n$. The known demand in period i is $d_i > 0$. The firm must satisfy demand in each period with stock on hand at that time. Let x_i be the amount produced in period i, and let z_i be the inventory on hand at the end of period i (and beginning of period $i+1$). Assume that there is no inventory on hand at the beginning of period 1 (so $z_0 = 0$) and that there is no inventory remaining at the end of the last period (so $z_n = 0$). There is a production cost $g_i(x_i)$ for producing x_i in period i and a holding cost $h_i(z_i)$ for inventory z_i held at the end of period i. The problem of the firm is to determine the level of production in each period so that total cost is minimized subject to the requirements that demand be satisfied in each period and no inventory remain after period n. The problem of the firm is to

$$\text{minimize } \textstyle\sum_{i=1}^{n} g_i(x_i) + \sum_{i=1}^{n} h_i(z_i)$$

subject to

$$z_i = z_{i-1} + x_i - d_i = z_0 + \textstyle\sum_{j=1}^{i} x_j - \sum_{j=1}^{i} d_j \quad \text{for } i = 1, \ldots, n,$$

$$z_0 = z_n = 0,$$

$$z_i \geq 0 \quad \text{for } i = 1, \ldots, n-1,$$

and

$$x_i \geq 0 \quad \text{for } i = 1, \ldots, n.$$

When the firm's cost functions are concave, Theorem 3.5.2 establishes the structure of optimal production vectors and shows how a suitable application of the recursion (3.5.1) can be used for the construction of optimal production vectors. Theorem 3.5.2 is based on Wagner and Whitin [1958] and Zangwill [1969]. For an n period problem, let $x = (x_1, \ldots, x_n)$ denote a vector of production levels in each period and let $z = (z_1, \ldots, z_n)$ denote the corresponding vector of inventory levels at the end of each period. For $i = 1, \ldots, n+1$, let $f(i)$ be the minimum total cost for the firm to engage in an $i-1$ period production problem with no inventory remaining at the end of period $i-1$ (that is, $z_{i-1}=0$). For $1 \le i' < i'' \le n+1$, let $c(i', i'')$ be the total cost in periods $i', \dots, i''-1$ given that $z_{i'-1} = z_{i''-1} = 0$, $x_{i'} = \sum_{i=i'}^{i''-1} d_i$, and $x_i = 0$ for $i = i'+1, \dots, i''-1$; that is,

$$c(i', i'') = g_{i'}\left(\textstyle\sum_{i=i'}^{i''-1} d_i\right) + \textstyle\sum_{i=i'+1}^{i''-1} g_i(0) + \textstyle\sum_{i=i'}^{i''-1} h_i\left(\textstyle\sum_{j=i+1}^{i''-1} d_j\right). \tag{3.5.2}$$

For $i = 2, \dots, n+1$, let $J(i)$ be the set of all i' that are optimal given i in (3.5.1) and given the particular values for $c(i', i'')$ as defined in (3.5.2). For $i = 1, \dots, n$, construct a production vector $x = (x_1, \dots, x_i)$ for the i period problem as follows. Pick any $j(i+1)$ in $J(i+1)$, and set $x_{j(i+1)} = \sum_{j=j(i+1)}^{i} d_j$ and $x_j = 0$ for $j = j(i+1)+1, \dots, i$. If $j(i+1) > 1$, then pick any $j(j(i+1))$ in $J(j(i+1))$, and set $x_{j(j(i+1))} = \sum_{j=j(j(i+1))}^{j(i+1)-1} d_j$ and $x_j = 0$ for $j = j(j(i+1))+1, \dots, j(i+1)-1$. Continue this procedure until a value is assigned to x_1.

An **extreme point** of a subset X of R^n is an element x' of X for which there do not exist distinct x'' and x''' in X and α with $0 < \alpha < 1$ such that $x' = \alpha x'' + (1-\alpha)x'''$.

Theorem 3.5.2. *Suppose that each function $g_i(x_i)$ and $h_i(z_i)$ is concave for $i = 1, \dots, n$.*
(a) *A feasible production vector x is an extreme point of the set of feasible production vectors if and only if $x_i > 0$ for any i implies that $z_{i-1} = 0$ and that $x_i = \sum_{j=i}^{i'} d_j$ for some i' with $i \le i' \le n$.*
(b) *There exists an optimal production vector that is an extreme point of the set of all feasible production vectors, and each such optimal production vector satisfies the conditions of part* (a)*.*
(c) *The value of each $f(i)$ can be computed from* (3.5.1) *with $c(i', i'')$ defined as in* (3.5.2)*.*
(d) *For $i = 1, \dots, n$, the production vector whose construction is based on* (3.5.1) *for the i period problem is optimal for the i period problem.*
(e) *For $i = 1, \dots, n$, $J(i+1)$ is the set of all i' such that there exists an optimal production vector* (*an optimal production vector that is an extreme point*) *for the i period problem, where i' is the last period having a positive production level.*

Proof. Let x be an extreme point of the set of feasible production vectors. Pick any i with $x_i > 0$. Suppose that $z_{i-1} > 0$, and let i' be the last period before period i in which production is positive. Let x' (respectively, x'') be a production vector that is identical to x except that x_i is replaced by $x_i + \epsilon$ (respectively, $x_i - \epsilon$) and $x_{i'}$ is replaced by $x_{i'} - \epsilon$ (respectively, $x_{i'} + \epsilon$) for some ϵ with $0 < \epsilon \le \min\{x_i, x_{i'}, z_{i-1}\}$. Then $x = (1/2)x' + (1/2)x''$ and both x' and

x'' are feasible production vectors. But this contradicts x being an extreme point, so $z_{i-1} = 0$. Now pick i'' such that period $i'' + 1$ is the first period after period i in which production is positive if any such period exists and $i'' = n$ otherwise. Thus, $x_j = 0$ for $j = i+1, \ldots, i''$. Just as $z_{i-1} = 0$ above, $z_{i''} = 0$. Therefore,

$$0 = z_{i''} = z_{i-1} + \sum_{j=i}^{i''} x_j - \sum_{j=i}^{i''} d_j = x_i - \sum_{j=i}^{i''} d_j,$$

and so $x_i = \sum_{j=i}^{i''} d_j$.

Let x be a feasible production vector for which $x_i > 0$ for any i implies that $z_{i-1} = 0$ and that $x_i = \sum_{j=i}^{i'} d_j$ for some i' with $i \leq i' \leq n$, where z is the vector of inventory levels corresponding to x. Suppose that x is not an extreme point of the set of feasible production vectors, so there exist distinct feasible production vectors x' and x'' and α with $0 < \alpha < 1$ such that $x = \alpha x' + (1-\alpha)x''$. Let z' and z'' be the vectors of inventory levels corresponding to x' and x'', respectively. Pick any i with $x_i > 0$. Let i' be the smallest $j \geq i$ with $x_{j+1} > 0$ if such j exists, and let $i' = n$ otherwise. Then $z_{i-1} = z_{i'} = x_{i+1} = \ldots = x_{i'} = 0$. Therefore, $z'_{i-1} = z'_{i'} = x'_{i+1} = \ldots = x'_{i'} = 0$ and $z''_{i-1} = z''_{i'} = x''_{i+1} = \ldots = x''_{i'} = 0$. Consequently, $x'_i = x''_i = \sum_{j=i}^{i'} d_j = x_i$ and so $x' = x'' = x$, which is a contradiction. Hence, x is an extreme point of the set of all feasible production vectors, completing the proof of part (a).

The problem of the firm here involves minimizing a concave function over a nonempty compact convex set, and so there exists an optimal production vector that is an extreme point of the set of feasible production vectors (Rockafellar [1970]). This, together with part (a), implies part (b).

Now consider an i period production problem for $1 \leq i \leq n$ with the requirement that $z_i = 0$. Consider some optimal production vector satisfying (by part (b)) the structure of part (a), and let i' be the last period in which production is positive. By part (a), $z_{i'-1} = 0$. The cost in periods $1, \ldots, i'-1$ is $f(i')$. The cost in periods $i', \ldots, i$ is $c(i', i+1)$. Hence, $f(i+1) = f(i') + c(i', i+1)$. Furthermore, for any i'' with $1 \leq i'' \leq i$ there exists a feasible production vector with a cost $f(i'') + c(i'', i+1)$ and so $f(i+1) \leq f(i'') + c(i'', i+1)$, concluding the proof of part (c).

The proof of part (d) proceeds by induction on i. The statement of part (d) holds for $i = 1$. Suppose that for some i' with $1 < i' \leq n$ the statement of part (d) holds for $i = 1, \ldots, i'-1$. It suffices to show that the statement of part (d) holds for $i = i'$. Let $(x_1, \ldots, x_{i'})$ be the production vector based on (3.5.1) for the i' period problem. By the induction hypothesis, $(x_1, \ldots, x_{j(i'+1)-1})$ is optimal for the $j(i'+1) - 1$ period problem and so the cost in periods $1, \ldots, j(i'+1) - 1$ given the production vector $(x_1, \ldots, x_{i'})$ is $f(j(i'+1))$. By construction, $z_{j(i'+1)-1} = z_{i'} = 0$, $(x_1, \ldots, x_{i'})$ is feasible for the i' period problem, and the cost in periods $j(i'+1), \ldots, i'$ given the production vector $(x_1, \ldots, x_{i'})$ is $c(j(i'+1), i'+1)$. Therefore, the cost in periods $1, \ldots, i'$ given the production vector $(x_1, \ldots, x_{i'})$ is $f(j(i'+1)) + c(j(i'+1), i'+1)$, which equals $f(i'+1)$ by part (c). This completes the proof of part (d).

By part (a) and part (d), if i' is in $J(i+1)$ then there exists an optimal production vector that is an extreme point for the i period problem where i' is the last period having a positive production level. By part (a) and part (c), if there exists an optimal production vector that is an extreme point for the i period problem where i' is the last period having a positive production level then i' is in $J(i+1)$. Because the problem of the firm involves minimizing a concave function over a nonempty compact convex set, it follows that if there exists an optimal production vector for the i period problem where i' is the last period having a positive production level then there exists an optimal production vector that is an extreme point for the i period problem where i' is the last period having a positive production level. This completes the proof of part (e). □

Corollary 3.5.1 combines results of Theorem 3.5.1 and Theorem 3.5.2 to elucidate the role of submodularity in structural properties of the dynamic economic lot size production model. In addition to the concavity assumptions of Theorem 3.5.2, Corollary 3.5.1 also assumes that the function $c(i', i'')$ as defined in (3.5.2) is submodular in (i', i''). Corollary 3.5.1 extends versions of results of Eppen, Gould, and Pashigian [1969], Wagner and Whitin [1958], and Zangwill [1969], who assume specific forms for the holding and production cost functions that implicitly imply that $c(i', i'')$ is submodular. The hypotheses of Corollary 3.5.1 hold if the inventory holding cost function $h_i(z_i)$ is concave and increasing in z_i for each period i and if the production cost function $g_i(x_i)$ consists of a fixed set-up cost a_i incurred whenever $x_i > 0$ plus a linear term $b_i x_i$ where a_i is nonnegative and b_i is decreasing in i. Under the hypotheses of Corollary 3.5.1, the last period in which it is optimal to have a positive level of production for an i period problem is increasing with i, and this property may be used to reduce the effort in computing optimal production vectors. Furthermore, the result gives sufficient conditions for a **planning horizon**, where if it is optimal to produce a positive amount in period i of an i period problem (that is, if i is in $J(i+1)$), then optimal production levels in periods $1, \ldots, i-1$ of the i period problem are optimal in periods $1, \ldots, i-1$ of the i' period problem for each $i' \geq i$ regardless of the specific values of the costs in periods $i+1, \ldots, i'$.

Corollary 3.5.1. *Suppose that each function $g_i(x_i)$ and $h_i(z_i)$ is concave for $i = 1, \ldots, n$ and that $c(i', i'')$ is submodular in (i', i'') on the sublattice $\{(i', i'') : 1 \leq i' < i'' \leq n+1,\ i' \text{ and } i'' \text{ integer}\}$ of R^2.*

(a) *The set $J(i+1)$, consisting of all periods i' such that there exists some optimal production vector for the i period problem for which period i' is the last period with a positive level of production, is increasing in i for i in $\{1, \ldots, n\}$.*
(b) *If there exists an optimal production vector for the $i-2$ period problem such that $j(i-1)$ is the last period with a positive level of production (that is, if $j(i-1)$ is in $J(i-1)$), then the domain of minimization $\{i' : 1 \leq i' \leq i-1\}$ in (3.5.1) for the $i-1$ period problem can be reduced to $\{i' : j(i-1) \leq i' \leq i-1\}$ (that is, there exists $j(i)$ in $J(i)$ with $j(i-1) \leq j(i)$).*
(c) *If $j(i+1)$ is the latest (or earliest) period for which it is ever optimal to have a positive level of production for the last time in an i period problem (that is, if $j(i+1)$ is the greatest (or least) element of $J(i+1)$), then $j(i+1)$ is increasing in i for i in $\{1, \ldots, n\}$.*
(d) *If for any period i there exists an optimal production vector for the i period problem such that there is a positive level of production in period i (that is, if i is in $J(i+1)$), then for each period i' with $i < i' \leq n$ there exists an optimal production vector for the i' period problem such that the level of production in each of periods $1, \ldots, i-1$ is the same as in any optimal production vector for the $i-1$ period problem.*

Proof. Part (a) follows from part (a) of Theorem 3.5.1 and part (e) of Theorem 3.5.2.

Part (b) follows from part (b) of Theorem 3.5.1 and part (c) of Theorem 3.5.2.

Part (c) follows from part (c) of Theorem 3.5.1 and part (e) of Theorem 3.5.2.

Part (d) follows from part (d) of Theorem 3.5.1 and part (c), part (d), and part (e) of Theorem 3.5.2. □

3.6 Production Planning

This section examines parametric properties of a deterministic multiperiod multiproduct production planning problem with respect to cumulative demands and with respect to bounds on the net inventory levels. Results give conditions for optimal cumulative production levels to increase with the parameters and for components of the parameters to be complementary. An earlier version of parts of this section is in Topkis [1968].

A firm produces m products $j = 1, \ldots, m$ in each of n consecutive periods $i = 1, \ldots, n$. For each period i and each product j, $d_{i,j}$ is the known demand for product j in period i. The firm determines the net production $x_{i,j}$ for each product j in each period i. There are an upper bound $b_{i,j}$ and a lower bound $a_{i,j}$ on the net production $x_{i,j}$ of product j in period i. Some $x_{i,j}$ may be negative (if $a_{i,j} < 0$), indicating that $-x_{i,j}$ units of product j inventory are disposed of (in some way other than to satisfy demand) in period i. The cumulative production of product j in the first i periods is required to be in a set $\mathcal{X}_{i,j}$. This last constraint could be used to model a requirement that product j can only be produced in multiples of some $\gamma_j > 0$ by letting $\mathcal{X}_{i,j}$ be a subset of $\{0, \gamma_j, 2\gamma_j, \dots\}$. Any demand that is not satisfied in some period is backlogged to the next period. No stock is held in inventory for any given product as long as some demand for that product is still unsatisfied. Assume, for convenience, that there exists neither any backlogged demand nor any stock on hand at the beginning of period 1. The net inventory level for product j at the end of period i is $z_{i,j} = \sum_{k=1}^{i} x_{k,j} - \sum_{k=1}^{i} d_{k,j}$, where $z_{i,j} \geq 0$ indicates an inventory level of $z_{i,j}$ and no backlogged demand for product j at the end of period i and where $z_{i,j} \leq 0$ indicates $-z_{i,j}$ units of backlogged demand and no inventory on hand for product j at the end of period i. There are an upper bound $u_{i,j}$ and a lower bound $v_{i,j}$ on the net inventory level $z_{i,j}$ for product j at the end of period i. (If $v_{i,j} \geq 0$, then no backlogging of demand for product j is permitted at the end of period i.) There is a production cost $c_{i,j}(x_{i,j})$ for a net production of $x_{i,j}$ units of product j in period i. There is a cost $h_i(z_{i,1}, \dots, z_{i,m})$ for ending period i with an m-vector of net inventory levels $(z_{i,1}, \dots, z_{i,m})$, where product j generates a holding cost for $z_{i,j}$ units of inventory if $z_{i,j} \geq 0$ and a penalty cost for $-z_{i,j}$ units of backlogged demand if $z_{i,j} \leq 0$.

The objective of the firm is to determine a production plan for all products in all periods so as to minimize the sum of all costs over all feasible production plans. This problem is to

$$\text{minimize } \sum_{j=1}^{m} \sum_{i=1}^{n} c_{i,j}(x_{i,j}) + \sum_{i=1}^{n} h_i(z_{i,1}, \dots, z_{i,m})$$

subject to

$$z_{i,j} = \sum_{k=1}^{i} x_{k,j} - \sum_{k=1}^{i} d_{k,j} \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m,$$

$$a_{i,j} \leq x_{i,j} \leq b_{i,j} \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m,$$

$$v_{i,j} \leq z_{i,j} \leq u_{i,j} \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m,$$

and

$$\sum_{k=1}^{i} x_{k,j} \in \mathcal{X}_{i,j} \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

It is convenient to explicitly consider cumulative production and cumulative demand and to express this and other notation more concisely. Let

$x_i = (x_{i,1}, \ldots, x_{i,m})$, $d_i = (d_{i,1}, \ldots, d_{i,m})$, and $z_i = (z_{i,1}, \ldots, z_{i,m})$ be the m-vectors of production levels, demands, and inventory levels, respectively, for the m products in period i. The m-vector of cumulative demand for the products in the first i periods is $D_i = \sum_{k=1}^{i} d_k$. Let $D = (D_1, \ldots, D_n)$ be the matrix consisting of the nm cumulative demands $D_{i,j}$. The cumulative production of product j in the first i periods is $X_{i,j} = \sum_{k=1}^{i} x_{k,j}$ and the m-vector of cumulative production for the products in the first i periods is $X_i = \sum_{k=1}^{i} x_k$. Let the m-vector $X_0 = 0$. The net production $x_{i,j}$ of product j in period i is $X_{i,j} - X_{i-1,j}$. Let $X = (X_1, \ldots, X_n)$ be the matrix consisting of the nm cumulative production levels $X_{i,j}$, and let $\mathscr{X} = \times_{i=1}^{n} \times_{j=1}^{m} \mathscr{X}_{i,j}$ be the direct product of the nm constraint sets for the cumulative production levels. Because $\mathscr{X}$ is a product set in R^{nm}, it is a sublattice of R^{nm}. The net inventory level $z_{i,j}$ for product j at the end of period i is $X_{i,j} - D_{i,j}$, and $z_i = X_i - D_i$. Denote the m-vectors of upper bounds and lower bounds on production and inventory in each period i as $b_i = (b_{i,1}, \ldots, b_{i,m})$, $a_i = (a_{i,1}, \ldots, a_{i,m})$, $u_i = (u_{i,1}, \ldots, u_{i,m})$, and $v_i = (v_{i,1}, \ldots, v_{i,m})$. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

The set $S_{u,v,D}$ of matrices X of cumulative production levels that are feasible given u, v, and D is

$$S_{u,v,D} = \{X : X \in \mathscr{X}, a_i \le X_i - X_{i-1} \le b_i \text{ for } i = 1, \ldots, n,$$
$$v + D \le X \le u + D\}.$$

Given X in $S_{u,v,D}$, the total cost for the firm is

$$f(X, D) = \sum_{j=1}^{m} \sum_{i=1}^{n} c_{i,j}(X_{i,j} - X_{i-1,j}) + \sum_{i=1}^{n} h_i(X_i - D_i).$$

The firm's problem of minimizing the sum of all costs over all feasible production plans is to

$$\text{minimize } f(X, D) \text{ subject to } X \text{ in } S_{u,v,D}.$$

Theorem 3.6.1 treats the cumulative demand vector D as fixed and gives conditions on the production cost functions $c_{i,j}(x_{i,j})$ and the combined holding and penalty cost functions $h_i(z_i)$ such that the optimal cumulative production levels X are increasing in the upper and lower bounds (u, v) on the inventory levels and such that (invoking Theorem 2.6.1) the components of the bounds (u, v) are complements with respect to the minimum total cost for the firm after optimizing over its cumulative production levels.

Theorem 3.6.1. *Suppose that the production cost function $c_{i,j}(x_{i,j})$ is convex and continuous in $x_{i,j}$ for all i and j, the combined holding and penalty cost function $h_i(z_i)$ is submodular and lower semicontinuous in z_i for each i, $\mathscr{X}_{i,j}$ is closed for all i and j, and the matrix D of cumulative demands is fixed.*

(a) *The set* $\text{argmin}_{X \in S_{u,v,D}} f(X, D)$ *of optimal cumulative production plans is increasing in* (u, v) *on* $\{(u, v) : S_{u,v,D}$ *is nonempty*$\}$.
(b) *The optimal cost for the firm* $\min_{X \in S_{u,v,D}} f(X, D)$ *is submodular in* (u, v) *on* $\{(u, v) : S_{u,v,D}$ *is nonempty*$\}$.

Proof. By Lemma 2.2.2 and part (b) of Example 2.2.7, $\{(X, u, v) : X \in S_{u,v,D}\}$ is a sublattice of R^{3nm}. By part (b) of Lemma 2.6.2 and part (b) of Lemma 2.6.1, $f(X, D)$ is submodular in X. Then part (a) follows from Theorem 2.8.2 and part (b) follows from Theorem 2.7.6. □

The following is an example for the submodular combined holding and penalty cost function $h_i(z_i)$ as in Theorem 3.6.1.

Example 3.6.1. Each product j has a volume w_j per unit held in inventory. In each period i there is a holding cost $H_i(y_i)$ that depends on the total volume $y_i = \sum_{j=1}^{m} w_j(z_{i,j} \vee 0)$ of inventory held at the end of period i. The inventory holding cost function $H_i(y_i)$ is concave on $[0, \infty)$, so there is a decreasing marginal cost for the volume of inventory held. (This concavity would permit a positive fixed charge to be associated with any positive volume of inventory held.) Furthermore, in each period i there is a unit penalty cost $p_{i,j}$ for each unit of backlogged demand of each product j. Then $h_i(z_i) = \sum_{j=1}^{m} p_{i,j}((-z_{i,j}) \vee 0) + H_i(\sum_{j=1}^{m} w_j(z_{i,j} \vee 0))$ is submodular in z_i by part (b) of Lemma 2.6.1 and part (a) of Lemma 2.6.2.

Theorem 3.6.2 considers parametric properties with the cumulative demand vector D. With the assumptions that $m = 1$ and that each combined holding and penalty cost function $h_i(z_i)$ is convex added to the hypotheses of Theorem 3.6.1, Theorem 3.6.2 shows that the optimal cumulative production levels X are increasing in the cumulative demands D and that (invoking Theorem 2.6.1), for all distinct periods i' and i'', the demand $D_{i',1}$ and the demand $D_{i'',1}$ are complements, the upper bound $u_{i',1}$ and the demand $D_{i'',1}$ are complements, and the lower bound $v_{i',1}$ and the demand $D_{i'',1}$ are complements. Using other methods, Veinott [1964] establishes a version of part (a) of Theorem 3.6.2 with a more restrictive ordering property.

Theorem 3.6.2. *Suppose that there is one product* $(m = 1)$*, the production cost function* $c_{i,1}(x_{i,1})$ *is convex and continuous in* $x_{i,1}$ *for each* i*, the combined holding and penalty cost function* $h_i(z_i)$ *is convex and continuous in* $z_i = z_{i,1}$ *for each* i*, and* $\mathscr{X}_{i,1}$ *is closed for each* i.
(a) *The set* $\text{argmin}_{X \in S_{u,v,D}} f(X, D)$ *of optimal cumulative production plans is increasing in* D *on* $\{D : S_{u,v,D}$ *is nonempty*$\}$.
(b) *The optimal cost for the firm* $\min_{X \in S_{u,v,D}} f(X, D)$ *is submodular in* D *on* $\{D : S_{u,v,D}$ *is nonempty*$\}$ *and submodular in* $(u_{i'}, v_{i'}, D_{i''}) = (u_{i',1}, v_{i',1}, D_{i'',1})$ *on* $\{(u_{i'}, v_{i'}, D_{i''}) : S_{u,v,D}$ *is nonempty*$\}$ *for all distinct periods* i' *and* i''.

Proof. By Lemma 2.2.2 and part (b) of Example 2.2.7, $\{(X, D) : X \in S_{u,v,D}\}$ is a sublattice of R^{2n} for fixed (u, v). By part (b) of Lemma 2.6.2 and part (b) of Lemma 2.6.1, $f(X, D)$ is submodular in (X, D). Then part (a) follows from Theorem 2.8.2 and the first part of part (b) follows from Theorem 2.7.6.

Now pick any distinct integers i' and i'' with $1 \le i' \le n$ and $1 \le i'' \le n$. Consider as variables the components $(u_{i',1}, v_{i',1}, D_{i'',1})$ of (u, v, D), and keep all other components of (u, v, D) fixed. By part (b) of Example 2.2.7, $\{(X, u_{i',1}, v_{i',1}, D_{i'',1}) : X \in S_{u,v,D}\}$ is a sublattice of R^{n+3} where the remaining $3n - 3$ components of (u, v, D) are fixed. By part (b) of Lemma 2.6.2 and part (b) of Lemma 2.6.1, $f(X, D)$ is submodular in $(X, u_{i',1}, v_{i',1}, D_{i'',1})$ on $\{(X, u_{i',1}, v_{i',1}, D_{i'',1}) : X \in S_{u,v,D}\}$. Then the second part of part (b) follows from Theorem 2.7.6. □

A proof similar to that of part (b) of Theorem 3.6.2 cannot be used to show for $m = 1$ that either u_i and D_i or v_i and D_i are either complements or substitutes (that is, that the optimal cost $\min_{X \in S_{u,v,D}} f(X, D)$ is submodular in (u_i, D_i), in (v_i, D_i), in $(u_i, -D_i)$, or in $(v_i, -D_i)$) because Lemma 2.2.7 shows that neither $\{(X, u_i, D_i) : X \in S_{u,v,D}\}$, $\{(X, v_i, D_i) : X \in S_{u,v,D}\}$, $\{(X, u_i, -D_i) : X \in S_{u,v,D}\}$, nor $\{(X, v_i, -D_i) : X \in S_{u,v,D}\}$ is generally a sublattice of R^{n+2} and so the sublattice hypothesis of Theorem 2.7.6 is not satisfied.

If each production cost function $c_{i,j}(x_{i,j})$ and each combined holding and penalty cost function $h_i(z_i)$ is convex and each set $\mathscr{X}_{i,j}$ is a convex set, then the property that the minimization operation preserves convexity (Dantzig [1955]) implies that the optimal cost $\min_{X \in S_{u,v,D}} f(X, D)$ is convex in (u, v, D) for any number of products m. Thus, the firm realizes increasing marginal costs with respect to each component of (u, v, D).

3.7 Minimum Cuts, Maximum Closures, and the Selection Problem

This section is concerned with a problem of optimally selecting activities for a firm. First, Subsection 3.7.1 and Subsection 3.7.2 analyze complementarities and monotone comparative statics for two related network problems, the minimum cut problem and the maximum closure problem, respectively. Certain combinatorial decision problems can be modeled with formulations that are cases of these network problems, and efficient algorithms are available for finding optimal solutions. Subsection 3.7.3 uses results from Subsection 3.7.1 and Subsection 3.7.2 to analyze the selection problem of a firm, where each activity selected yields an intrinsic (nonnegative or nonpositive) return and where the presence of shared fixed costs may make it costly to select certain activities without also selecting certain other activities. The activities and their returns are shown to be complementary, and optimal selections of activities increase with the individual returns and with the set of available activities. Subsection 3.7.4 shows that the collections of solutions for the minimum cut problem and the maximum closure problem are structurally not different, in general, from an arbitrary sublattice of the power set of a finite set or from the set of maxima of an arbitrary supermodular function on a sublattice of the power set of a finite set.

3.7.1 Minimum Cut Problem

Consider a directed network, consisting of a finite set N of nodes and a directed edge joining each ordered pair of nodes. There is a nonnegative real-valued capacity $c(x', x'')$ on the edge from any node x' to any node x''. This is a **capacitated network**. (There is no loss of generality in assuming that there is an edge joining each ordered pair of nodes, because an edge can be effectively eliminated by setting its capacity to 0.) Two specific distinct nodes s and t in N are designated the **source** and **sink**, respectively. If X is a subset of nodes containing the source but not the sink, then X is a **cut**. The union and the intersection of any two cuts are also cuts, so the collection of all cuts is a sublattice of the power set $\mathcal{P}(N)$ of the set of nodes N and is itself a lattice. (The power set $\mathcal{P}(N)$ consisting of all subsets of the set of nodes N is taken to have the set inclusion ordering relation $\subseteq$.) If X' and X'' are sets of nodes, then let $(X', X'') = \{(x', x'') : x' \in X', x'' \in X''\}$ denote the collection of all edges from some node in X' to some node in X''. (A **cut** is sometimes defined as the set of edges $(X, N \setminus X)$ where X is a cut as defined above, but herein a cut refers to a set of nodes rather than the corresponding set of edges.) For subsets X' and X'' of N, let $c(X', X'') = \sum_{x' \in X', x'' \in X''} c(x', x'')$ be the sum of all capacities on all edges of (X', X''). For any cut X, let $f(X) = c(X, N \setminus X)$. The function $f(X)$ is the **cut capacity function**, as it measures the capacity leaving any cut X. (One could define $f(X)$ for each X in $\mathcal{P}(N)$ and Lemma 3.7.1 would hold on this larger domain, but problems of present interest involve properties of $f(X)$ on the collection of all cuts.)

The **minimum cut problem** is to minimize $f(X)$ subject to X being a cut. An optimal solution for this problem is a **minimum cut**. The minimum cut problem is the dual of the maximum flow problem for which efficient algorithms are available, and after finding a maximum flow it is a simple matter to find the least minimum cut and the greatest minimum cut (Ahuja, Magnanti, and Orlin [1993]; Ford and Fulkerson [1962]). Lemma 3.7.1 shows the well-known property that the cut capacity function is submodular.

Lemma 3.7.1. *The cut capacity function $f(X)$ is submodular in X for X being a cut.*

Proof. Pick any cuts X' and X''. Then

$$\begin{aligned}
&f(X') + f(X'') - f(X' \cup X'') - f(X' \cap X'') \\
&\quad = c(X', N \setminus X') + c(X'', N \setminus X'') - c(X' \cup X'', N \setminus (X' \cup X'')) \\
&\qquad - c(X' \cap X'', N \setminus (X' \cap X'')) \\
&\quad = c(X', N \setminus X') + c(X'', N \setminus X'') \\
&\qquad - c(X', N \setminus (X' \cup X'')) - c(X'', N \setminus (X' \cup X'')) \\
&\qquad + c(X' \cap X'', N \setminus (X' \cup X'')) - c(X' \cap X'', N \setminus (X' \cap X'')) \\
&\quad = c(X', X'' \setminus X') + c(X'', X' \setminus X'') \\
&\qquad - c(X' \cap X'', X'' \setminus X') - c(X' \cap X'', X' \setminus X'') \\
&\quad = c(X' \setminus X'', X'' \setminus X') + c(X'' \setminus X', X' \setminus X'') \geq 0. \quad \square
\end{aligned}$$

Ford and Fulkerson [1962] and Shapley [1961] show that the union and the intersection of any two minimum cuts in a capacitated network are also minimum cuts. That conclusion also follows as a direct consequence of applying Theorem 2.7.1 to Lemma 3.7.1. Furthermore, by part (a) of Corollary 2.7.1, there are a greatest minimum cut and a least minimum cut.

Let $c(s) = \{c(s, x) : x \in N\}$ and $c(t) = \{c(x, t) : x \in N\}$ denote, respectively, the vector of capacities of all edges leaving the source s and the vector of capacities of all edges entering the sink t. Subsequent issues involve how minimum cuts and the optimal value of the objective function in the minimum cut problem depend on the capacities $c(s)$ and $c(t)$. Define the parameterized cut capacity function $f(X, c(s), c(t))$ to be the value of the cut capacity function $f(X)$ for any cut X where the function depends explicitly on the capacity vectors $c(s)$ and $c(t)$ as parameters. Lemma 3.7.2 extends Lemma 3.7.1 by showing that the parameterized cut capacity function $f(X, c(s), c(t))$ is submodular in $(X, c(s), -c(t))$.

Lemma 3.7.2. *The parameterized cut capacity function $f(X, c(s), c(t))$ is submodular in $(X, c(s), -c(t))$ for X being a cut, $c(s) \geq 0$, and $c(t) \geq 0$.*

Proof. The proof is based on applying Theorem 2.6.2. Using the observation of part (d) of Example 2.2.1, the domain of the parameterized cut capacity function $f(X, c(s), c(t))$ can be taken as the direct product of $3|N|$ chains. Because

$$\begin{aligned} f(X, c(s), c(t)) &= c(X, N \setminus X) \\ &= c(\{s\}, N \setminus X) + c(X \setminus \{s\}, N \setminus (X \cup \{t\})) \\ &\quad + c(X \setminus \{s\}, \{t\}) \\ &= c(\{s\}, N) - c(\{s\}, X) + c(X \setminus \{s\}, N \setminus (X \cup \{t\})) \\ &\quad + c(X \setminus \{s\}, \{t\}), \end{aligned}$$

$f(X, c(s), c(t))$ has decreasing differences in $(X, c(s))$ and decreasing differences in $(X, -c(t))$ where X is restricted to be a cut. Furthermore, $f(X, c(s), c(t))$ is submodular in X by Lemma 3.7.1 and $f(X, c(s), c(t))$ is separable (and hence submodular) in the components of $(c(s), -c(t))$. Therefore, the conditions of Theorem 2.6.2 hold and $f(X, c(s), c(t))$ is submodular in $(X, c(s), -c(t))$. □

The statement of Theorem 3.7.1, that the minimum cut capacity in the minimum cut problem is submodular in $(c(s), -c(t))$, follows from Lemma 3.7.2 and Theorem 2.7.6. Consequences of Theorem 3.7.1 together with Theorem 2.6.1 are that the capacities on any two edges leaving the source are substitutes, the capacities on any two edges entering the sink are substitutes, and the capacities on any edge leaving the source and on any edge entering the sink are complements. (The definitions of "complement" and "substitute" from Subsection 2.6 are here interpreted in terms of the dual maximization problem, the maximum flow problem, rather than the minimization problem, the minimum cut problem.) Shapley [1961] shows these substitutability and complementarity properties by other methods.

Theorem 3.7.1. *The optimal value of the objective function in the minimum cut problem (that is, the minimum of $f(X, c(s), c(t))$ over all cuts X) is submodular in $(c(s), -c(t))$.*

Theorem 3.7.2 follows from Lemma 3.7.2 and Theorem 2.8.2. By Theorem 3.7.2, minimum cuts increase with larger capacities on edges leaving the source s and with smaller capacities on edges entering the sink t.

Theorem 3.7.2. *The collection of minimum cuts $\text{argmin}_{\{X: X \text{ is a cut}\}} f(X, c(s), c(t))$ is increasing in $c(s)$ and decreasing in $c(t)$ for $c(s)$ and $c(t)$ nonnegative.*

Theorem 3.7.2 implicitly shows how minimum cuts may vary as nodes with certain properties are added to or deleted from a capacitated network, as noted in Corollary 3.7.1. In particular, Corollary 3.7.1 applies to parametrically adding or deleting a node directly following the source or directly preceding the sink in an acyclic network.

Corollary 3.7.1. *Consider a capacitated network.*
(a) *If x' is a node other than the source or the sink and $c(x, x') = 0$ for each node x other than the source, then the collection of minimum cuts decreases as x' is deleted from the network.*
(b) *If x' is a node other than the source or the sink and $c(x', x) = 0$ for each node x other than the sink, then the collection of minimum cuts increases as x' is deleted from the network.*

Proof. Instead of deleting node x' as in part (a), consider increasing the edge capacity $c(x', t)$ from its actual value to a value larger than the sum of all other edge capacities. With this change, the node x' is not in any minimum cut and the collection of minimum cuts is the same as in the capacitated network with x' deleted because $c(x, x') = 0$ for each node x other than the source. Furthermore, the increase in $c(x', t)$ decreases the collection of minimum cuts by Theorem 3.7.2, completing the proof of part (a).

Instead of deleting node x' as in part (b), consider increasing the edge capacity $c(s, x')$ from its actual value to a value larger than the sum of all other edge capacities. With this change, the node x' is included in every minimum cut and the collection of minimum cuts modified by the deletion of x' from each minimum cut is the same as in the capacitated network with x' deleted because $c(x', x) = 0$ for each node x other than the sink. Furthermore, the increase in $c(s, x')$ increases the collection of minimum cuts by Theorem 3.7.2, completing the proof of part (b). □

3.7.2 Maximum Closure Problem

Consider a directed network with a set of nodes N', a set of directed edges A that is a subset of $N' \times N'$, and a real weight $w(x)$ associated with each node x. A **closure** is a subset X of the nodes N' such that if x' is in X and the edge (x', x'') is in A then x'' is in X. The union and the intersection of any two closures are also closures, and so the collection of all closures is a sublattice of the power set $\mathcal{P}(N')$. For each subset X of the nodes N', let $w(X) = \sum_{x \in X} w(x)$ be the sum of the weights associated with the nodes in X. The **maximum closure problem** is to maximize $w(X)$ over all closures X. An optimal solution for this problem is a **maximum closure**.

Picard [1976] shows that the maximum closure problem can be solved by solving a corresponding minimum cut problem, as follows. Suppose that one is given a maximum closure problem with nodes N', directed edges A, and a weight function $w(x)$ on N'. Let γ be any real number with $\sum_{x \in N'} |w(x)| < \gamma$. Construct a corresponding capacitated network with nodes $N = N' \cup \{s, t\}$ where s is the source and t is the sink, with a directed edge joining each ordered pair of nodes, and with edge capacities on $N \times N$ such that $c(s, x) = w(x)$ if x is in N' and $w(x) > 0$, $c(x, t) = -w(x)$ if x is in N' and $w(x) < 0$, $c(x', x'') = \gamma$ if (x', x'') is in A, and each other edge capacity is 0. Lemma 3.7.3, from Picard [1976], establishes direct connections between solutions for the maximum closure problem and solutions for the minimum cut problem in the corresponding capacitated network.

Lemma 3.7.3. *Suppose that X is a subset of N'.*
(a) *The set X is a closure if and only if the capacity of the cut $X \cup \{s\}$ is less than γ (that is, $c(X \cup \{s\}, N \setminus (X \cup \{s\})) < \gamma$) in the corresponding capacitated network.*
(b) *If X is a closure, then $c(X \cup \{s\}, N \setminus (X \cup \{s\})) = \sum_{\{x : x \in N', w(x) > 0\}} w(x) - w(X)$.*
(c) *The set X is a maximum closure if and only if $X \cup \{s\}$ is a minimum cut in the corresponding capacitated network.*

Proof. The sum of all those edge capacities that are strictly less than γ is $\sum_{x \in N'} |w(x)| < \gamma$, and $c(x', x'') = \gamma$ if and only if (x', x'') is in A. Therefore, $c(X \cup \{s\}, N \setminus (X \cup \{s\})) < \gamma$ if and only if there is no (x', x'') in A with x' in X and x'' not in X. The latter condition is equivalent to X being a closure, which completes the proof of part (a).

If X is a closure, then

$$\begin{aligned} c(X \cup \{s\}, N \setminus (X \cup \{s\})) &= c(\{s\}, N' \setminus X) + c(X, \{t\}) \\ &= \textstyle\sum_{\{x : x \in N' \setminus X, w(x) > 0\}} w(x) + \sum_{\{x : x \in X, w(x) < 0\}} (-w(x)) \\ &= \textstyle\sum_{\{x : x \in N', w(x) > 0\}} w(x) - w(X) \end{aligned}$$

and so part (b) holds.

Part (c) follows from part (a) and part (b). □

Theorem 3.7.3 follows from Theorem 3.7.1 and part (b) of Lemma 3.7.3. By Theorem 3.7.3 together with Theorem 2.6.1, the components of $\{w(x) : x \in N'\}$ are complements; that is, the increase in the total weight of a maximum closure due to an increase in the value of any particular node weight increases with the values of the weights of the other nodes.

Theorem 3.7.3. *The maximum value of the weight function $w(X)$ over all closures X is supermodular as a function of the vector of weights $\{w(x) : x \in N'\}$.*

Theorem 3.7.4 follows from Theorem 3.7.2 and part (c) of Lemma 3.7.3. Maximum closures increase with the node weights.

Theorem 3.7.4. *The collection of maximum closures is increasing with the vector of weights $\{w(x) : x \in N'\}$.*

3.7.3 Selection Problem

Consider a firm that has a finite set N' of activities in which it may engage. The firm must decide which subset of the activities N' to select. For each activity x in N', the firm receives a net return $r(x)$ if it selects activity x. If $r(x) > 0$, then activity x is itself profitable for the firm. If $r(x) < 0$, then activity x is itself costly for the firm. Engaging in some activities may be facilitated if the firm also engages in certain other activities, and this is reflected in there being a cost $b(x', x'') \geq 0$ if the firm selects activity x' but does not select activity x'' for any x' and x'' in N'. The firm wants to select a subset of activities to maximize its profit. The problem of the firm is then to

$$\text{maximize } F(X) = \textstyle\sum_{x \in X} r(x) - \sum_{x' \in X, x'' \in N' \setminus X} b(x', x'') \text{ subject to } X \subseteq N',$$

where X is the set of activities that the firm selects from N'. This is the **selection problem**.

The selection problem has features of both the maximum closure problem (a value associated with each element that is selected) and the minimum cut problem (a nonnegative cost associated with any ordered pair of elements if the first element is selected and the second element is not selected). The selection problem can be modeled as a corresponding minimum cut problem as follows. The network has a node x associated with each activity x in N' as well as an additional source node s and an additional sink node t. There is a directed edge joining each ordered pair of nodes from $N' \cup \{s, t\}$, and there is a nonnegative capacity defined on each edge. The capacity on the edge from the source s to any node x in N' with $r(x) > 0$ is $r(x)$. The capacity on the edge from any node x in N' with $r(x) < 0$ to the sink t is $-r(x)$. The capacity on the edge from any node x' in N' to any node x'' in N' is $b(x', x'')$. All other edges have capacity 0. For any subset X of N', the capacity of the cut $X \cup \{s\}$ is

$$\begin{aligned} &\textstyle\sum_{\{x : x \in N' \setminus X, r(x) > 0\}} r(x) \\ &\quad + \textstyle\sum_{\{x : x \in X, r(x) < 0\}} (-r(x)) + \sum_{x' \in X, x'' \in N' \setminus X} b(x', x'') \\ &\quad = \textstyle\sum_{\{x : x \in N', r(x) > 0\}} r(x) - F(X). \end{aligned} \tag{3.7.1}$$

Equation (3.7.1) together with Lemma 3.7.1 implies that F(X) is supermodular in X, as in Lemma 3.7.4, so the activities are complementary by Theorem 2.6.1. Adding any activity to a selection causes a greater net increase in the firm's profit if the selection is initially modified by the addition of any other activity. Equation (3.7.1) also yields Lemma 3.7.5, which relates solutions for the selection problem and for the corresponding minimum cut problem.

Lemma 3.7.4. *The profit function $F(X)$ of the firm for the selection problem is supermodular in X for X being a subset of N'.*

Lemma 3.7.5. *If X is any subset of the activities N', then X is an optimal selection for the firm if and only if $X \cup \{s\}$ is a minimum cut in the corresponding capacitated network. Furthermore, the optimal profit for the firm equals $\sum_{\{x:x\in N',r(x)>0\}} r(x)$ minus the capacity of the minimum capacity cut in the corresponding capacitated network.*

Theorem 3.7.5 follows from Lemma 3.7.5 and Theorem 3.7.1, showing by Theorem 2.6.1 that the net returns of the various activities are complements. That is, the increase in the optimal profit to the firm resulting from an increase in the return $r(x')$ for any activity x' increases with the return $r(x'')$ for any other activity x''.

Theorem 3.7.5. *The maximum profit for the firm is a supermodular function of the vector $\{r(x) : x \in N'\}$ of returns for selecting the individual activities in N'.*

Theorem 3.7.6 follows from Lemma 3.7.5 and Theorem 3.7.2, showing that the sets of activities optimal for the firm increase with each component of $\{r(x) : x \in N'\}$.

Theorem 3.7.6. *The collection of optimal selections for the firm* $\text{argmax}_{X\subseteq N'} F(X)$ *is increasing in* $\{r(x) : x \in N'\}$.

If the set of feasible activities for the firm is limited to some subset N'' of N', then the optimal profit for the firm $\max_{X\subseteq N''} F(X)$ is a supermodular function of N'' for subsets N'' of N' by Lemma 3.7.4 and Theorem 2.7.6 and the collection of optimal selections for the firm $\text{argmax}_{X\subseteq N''} F(X)$ is increasing in N'' for subsets N'' of N' by Lemma 3.7.4 and Theorem 2.8.1.

The selection problem may also include requirements, similar to the maximum closure problem, that certain activities can be selected only if certain other activities are also selected. This requirement can be represented by a subset W of $N' \times N'$, where if (x', x'') is in W then the firm can select activity x' only if it also selects activity x''. Such requirements can be incorporated into the above selection problem by giving the cost $b(x', x'')$ a value strictly greater than $\sum_{\{x:x\in N',r(x)>0\}} r(x)$ for each (x', x'') in W. The revised selection problem with these modified values of $b(x', x'')$ for (x', x'') in W is equivalent to the original selection problem with the additional requirement that the selection of x' requires the selection of x'' for each (x', x'') in W.

A special case of this latter problem is studied by Balinski [1970] and Rhys [1970], who consider a version of the selection problem where each activity is either a project to be carried out or a facility which may be needed for one or more projects. Each project is in itself profitable, so the net return $r(x)$ from each project x is nonnegative. Each facility is in itself costly, so the net return $r(x)$ (that is, minus the cost) from each facility x is nonpositive. Activity x'' is required for activity x' (that is, (x', x'') is in W) only if activity x' is a project and activity x'' is a facility. The set of facilities required in order to carry out project x' is $\{x : (x', x) \in W\}$. The problem is to choose a set of facilities to maximize the value of the projects realizable with those facilities minus the cost of the facilities. The corresponding capacitated network without the source and sink is **bipartite**, since the nodes can be partitioned into two disjoint sets and any edge with positive capacity must be directed from a node in the first set (whose elements are associated with the projects) to a node in the second set (whose elements are associated with the facilities). Picard [1976] notes that this special case of the selection problem can be expressed as a maximum closure problem, and Picard's [1976] solution of the maximum closure problem as a minimum cut problem is a generalization of the result of Balinski [1970] and Rhys [1970] that one can find solutions for this special case of the selection problem by solving a minimum cut problem.

3.7.4 Equivalent Combinatorial Structures

Theorem 3.7.7 shows structural equivalences between a sublattice of the power set of a finite set, the collection of maximizers of a supermodular function on a sublattice of the power set of a finite set, the collection of minimum cuts in a capacitated network, and the collection of maximum closures in a network. This statement is from Monma and Topkis [1982]. Most of the implications constituting the proof of Theorem 3.7.7 have long been known, but the overall equivalence does not seem to have been stated previously. Gusfield and Irving [1989] examine related structural properties.

Theorem 3.7.7. *Suppose that N is a finite set with $|N| \ge 2$ and $\mathscr{X}$ is a collection of subsets of N. The following are equivalent.*
(a) *The collection $\mathscr{X}$ is a nonempty sublattice of $\mathscr{P}(N)$.*
(b) *There is a supermodular function $f(X)$ on $\mathscr{P}(N)$ such that $\mathscr{X} = \text{argmax}_{X\in\mathscr{P}(N)} f(X)$.*

(c) *There is a capacitated network with a source s, a sink t, and other nodes N such that $\mathcal{X}$ is the collection of all minimum cuts with the source s deleted from each.*
(d) *There is maximum closure problem in a network with nodes N such that $\mathcal{X}$ is the collection of maximum closures.*

Proof. By Theorem 2.7.1, if (b) holds then (a) holds. By Lemma 3.7.1, if (c) holds then (b) holds. By part (c) of Lemma 3.7.3, if (d) holds then (c) holds.

To complete the proof, it suffices to show that (a) implies (d). Suppose that $\mathcal{X}$ is a nonempty sublattice of $\mathcal{P}(N)$. For each x in $\cup_{X\in\mathcal{X}}X$, define $B(x) = \cap\{X : X \in \mathcal{X}, x \in X\}$. Because $\mathcal{X}$ is a nonempty sublattice of $\mathcal{P}(N)$, $\cup_{X\in\mathcal{X}}X$ is in $\mathcal{X}$, $\cap_{X\in\mathcal{X}}X$ is in $\mathcal{X}$, and $B(x)$ is in $\mathcal{X}$ for each x in $\cup_{X\in\mathcal{X}}X$. Construct a maximum closure problem as a network with a set of nodes N, a directed edge from node x' to node x'' if x' is in $\cup_{X\in\mathcal{X}}X$ and x'' is in $B(x') \setminus \{x'\}$, a weight $w(x) = 1$ associated with each node x in $\cap_{X\in\mathcal{X}}X$, a weight $w(x) = -1$ associated with each node x in $N \setminus (\cup_{X\in\mathcal{X}}X)$, and a weight $w(x) = 0$ associated with each node x in $(\cup_{X\in\mathcal{X}}X) \setminus (\cap_{X\in\mathcal{X}}X)$. Observe that $\cap_{X\in\mathcal{X}}X$ is a closure in the network and the total weight of any maximum closure is $|\cap_{X\in\mathcal{X}}X|$. Pick any X' in $\mathcal{X}$. By construction, X' is a closure in the network and its total weight is $w(X') = |\cap_{X\in\mathcal{X}} X|$ and so X' is a maximum closure. Now pick any maximum closure X''. Because X'' is a maximum closure, its weight is $|\cap_{X\in\mathcal{X}} X|$ and so $\cap_{X\in\mathcal{X}}X$ is a subset of X'' and X'' is a subset of $\cup_{X\in\mathcal{X}}X$. Because X'' is a closure, $B(x)$ is subset of X'' for each x in X'' and so $\cup_{x\in X''}B(x)$ is a subset of X''. But X'' is a subset of $\cup_{x\in X''}B(x)$, so $X'' = \cup_{x\in X''}B(x)$. Because $B(x)$ is in $\mathcal{X}$ for each x in $\cup_{X\in\mathcal{X}}X$, $\mathcal{X}$ being a sublattice of $\mathcal{P}(N)$ implies that $X'' = \cup_{x\in X''}B(x)$ is in $\mathcal{X}$. Therefore, the collection $\mathcal{X}$ is identical with the collection of maximum closures. □

3.8 Myopic Decisions

A firm engaged in a series of activities over multiple time periods may choose to myopically optimize its decisions in each individual period (that is, to maximize the net profit attributable to each period alone over the feasible decision variables for that period), without carefully taking into account the impact on net profits in subsequent periods of decisions made for any given period. Myopic decisions for a firm may be motivated by bounded rationality (e.g., the difficulty of accurately estimating parameters for future periods and the difficulty of optimizing over multiple time periods), by intrinsic separations between the effects of decisions in successive periods (as with a manufacturing system designed to maintain low inventory levels so production decisions in one period do not significantly impact the next period by the quantity of inventory held over), or by theoretical considerations (as discussed in this section) indicating that myopic decisions for the current period are optimal for the multiperiod problem. Subsection 3.8.1 develops general conditions under which myopic decisions for a firm are optimal in the long run. Some of the main ideas for the present conditions are implicit in the approach of Karlin [1960] for a single product stochastic inventory problem. Veinott [1965], with a correction attributed to Lowell Anderson (Veinott [1970]), considers myopic decisions for more general stochastic inventory problems. Heyman and Sobel [1984] discuss a variety of models having optimal myopic decisions. Subsection 3.8.2 gives conditions for the existence of optimal myopic solutions in a dynamic version of the selection problem and for these optimal myopic solutions to be increasing from period to period.

Assume throughout this section that, when needed, all real-valued functions are Borel measurable and integrable with respect to appropriate distribution functions.

3.8.1 General Conditions

A firm seeks to maximize the present value of its total profits over m successive periods $i = 1, \dots, m$. The firm's decision variable in each period i is an n-vector x_i which is constrained to be in a subset X_i of R^n. A second constraint is that in each period $i = 2, \dots, m$ a lower bound for the decision variable x_i is the realization of a random variable $z_i(x_{i-1})$ that depends on the decision variable x_{i-1} determined for period $i - 1$. An initial condition is a given lower bound z_1 for the decision variable x_1 in period 1. If the firm makes a feasible decision x_i in period i, then it receives an undiscounted (expected) net profit $f_i(x_i)$ in that period. There is a discount factor $\alpha_i \geq 0$ in each period i such that each unit of undiscounted net profit in that period has a value α_i in period 1. (One could state this model without explicitly including α_i, by replacing $f_i(x_i)$ with $\alpha_i f_i(x_i)$ so that $f_i(x_i)$ would then represent the (discounted) present value in period 1 of the net profit in period i. However, this would be inconvenient for stating the conditions of Theorem 3.8.1 below because the supermodularity of $f_i(x)$ in (x, i) does not imply the supermodularity of $\alpha_i f_i(x)$ in (x, i) and only the former condition is needed. Nevertheless, the dynamic selection problem example of Subsection 3.8.2 shows that even with the present model the discount factors may impact the functions $f_i(x_i)$ in formulations of particular problems.)

To formulate the problem of the firm as an m period dynamic programming problem, let $g_i(x_i)$ be the maximum (expected) present value in period 1 of total profits in periods $i, \dots, m$ given the decision x_i in period i. The values for each $g_i(x_i)$ can be computed recursively from

$$g_i(x_i) = \alpha_i f_i(x_i) + E(\max\{g_{i+1}(x_{i+1}) : x_{i+1} \in X_{i+1},\, z_{i+1}(x_i) \le x_{i+1}\}) \quad \text{for } i = 1, \ldots, m-1$$

and

$$g_m(x_m) = \alpha_m f_m(x_m),$$

where the expectation is taken with respect to the random variable $z_{i+1}(x_i)$ given x_i. The collection of optimal decisions x_i given $z_i(x_{i-1})$ is $\operatorname{argmax}_{\{x_i : x_i \in X_i, z_i(x_{i-1}) \le x_i\}} g_i(x_i)$ for $i = 2, \ldots, m$, and the collection of optimal decisions x_1 is $\operatorname{argmax}_{\{x_1 : x_1 \in X_1, z_1 \le x_1\}} g_1(x_1)$. If each lower bound $z_i(x_{i-1})$ is deterministic given x_{i-1}, then the problem of the firm is to

$$\text{maximize } \textstyle\sum_{i=1}^m \alpha_i f_i(x_i)$$

$$\text{subject to } x_i \text{ in } X_i \text{ for } i = 1, \ldots, m,$$

$$z_i(x_{i-1}) \le x_i \text{ for } i = 2, \ldots, m, \text{ and } z_1 \le x_1.$$

Decisions for the firm are **myopic** if in each period the decision variables for that period are determined by optimizing the profits in that period alone as a single period problem without regard for how the decision in that period may impact profits in any subsequent period; that is, $x_1, \ldots, x_m$ represent myopic decisions for the firm if x_i is in $\operatorname{argmax}_{x \in X_i} f_i(x)$ for each i. The issue considered here involves the possibility of myopic decisions being optimal for the firm. Because no feasible decision in any period can result in a greater profit for that period than a myopic decision, myopic decisions are optimal if they are feasible. The issue now involves conditions for myopic decisions to be feasible. Decisions for the firm in different periods are linked only through the lower bound constraints $z_i(x_{i-1}) \le x_i$ for $i = 2, \ldots, m$ as well as $z_1 \le x_1$ (which links the decision in period 1 with the initial condition z_1, and is categorized with the other lower bound constraints because it is exogenous to the period 1 problem). Thus myopic decisions $x_1, \ldots, x_m$ are optimal if and only if $z_1 \le x_1$ and $z_i(x_{i-1}) \le x_i$ for $i = 2, \ldots, m$ and each realization of the random variable $z_i(x_{i-1})$. The first of these conditions calls for $[z_1, \infty) \cap \operatorname{argmax}_{x \in X_1} f_1(x)$ to be nonempty, and the second condition calls for $[z_i(x_{i-1}), \infty) \cap \operatorname{argmax}_{x \in X_i} f_i(x)$ to be nonempty for $i = 2, \ldots, m$ and each possible realization of $z_i(x_{i-1})$. The latter property would hold if $z_i(x_{i-1}) \le x_{i-1}$ for $i = 2, \ldots, m$ and each possible realization of $z_i(x_{i-1})$ and if $[x_{i-1}, \infty) \cap \operatorname{argmax}_{x \in X_i} f_i(x)$ is nonempty for $i = 2, \ldots, m$. This last property would hold if $\operatorname{argmax}_{x \in X_i} f_i(x)$ is increasing in i for $i = 1, \ldots, m$, which corresponds to the conclusion of the statement of Theorem 2.8.2 and leads to the hypotheses and proof of Theorem 3.8.1. The additional regularity conditions (nonemptiness and either finiteness or compactness and upper semicontinuity) are included so that myopic decisions exist, part (a) of Theorem 2.8.3 applies, and there exist greatest myopic decisions in each period. By Theorem 2.4.5, the assumption of Theorem 3.8.1 that X_i is increasing in i for $i = 1, \ldots, m$ is equivalent to assuming that $\{(x, i) : x \in X_i \text{ and } i = 1, \ldots, m\}$ is a sublattice of R^{n+1} and that each X_i is nonempty.

Theorem 3.8.1. *If* $[z_1, \infty) \cap \text{argmax}_{x \in X_1} f_1(x)$ *is nonempty,* $z_i(x_{i-1}) \leq x_{i-1}$ *for each* x_{i-1} *in* X_{i-1} *and each possible realization of* $z_i(x_{i-1})$ *for* $i = 2, \ldots, m$, X_i *is increasing in* i *for* $i = 1, \ldots, m$, $f_i(x)$ *is supermodular in* (x, i) *on* $\{(x, i) : x \in X_i$ *and* $i = 1, \ldots, m\}$, *and either* X_i *is finite for each* i *or* X_i *is compact and* $f_i(x)$ *is upper semicontinuous in* x *on* X_i *for each* i, *then optimal myopic decisions exist for the firm. In particular, there exists a greatest myopic decision for each period, and these decisions are optimal for the firm.*

Proof. By part (a) of Theorem 2.8.3, there exists a greatest myopic decision x'_i (that is, a greatest element of $\text{argmax}_{x \in X_i} f_i(x)$) for $i = 1, \ldots, m$ with x'_i increasing in i. Let $x''_1, \ldots, x''_m$ be feasible decisions for the firm given the lower bound z_1 and any possible realizations for the lower bounds $z_2(x''_1), \ldots, z_m(x''_{m-1})$. Because x'_i is in $\text{argmax}_{x \in X_i} f_i(x)$ and x''_i is in X_i, $f_i(x''_i) \leq f_i(x'_i)$ for $i = 1, \ldots m$ and $\sum_{i=1}^m \alpha_i f_i(x''_i) \leq \sum_{i=1}^m \alpha_i f_i(x'_i)$. Therefore, the myopic decisions $x'_1, \ldots, x'_m$ are optimal if they are feasible. By hypothesis, $z_1 \leq x'_1$. By hypothesis and because x'_i is increasing in i, $z_i(x'_{i-1}) \leq x'_{i-1} \leq x'_i$ for each possible realization of $z_i(x'_{i-1})$ for $i = 2, \ldots, m$. The greatest myopic decisions are therefore feasible and hence optimal. □

While greatest myopic decisions for each period are optimal for the firm under the hypotheses of Theorem 3.8.1, the least myopic decisions may not be optimal because the lower bound z_1 in period 1 could cause infeasibility. When the least optimal decision in period 1 satisfies the lower bound z_1, the least myopic decisions for each period are also optimal for the firm.

Instead of defining myopic decisions as above (as any x_i in $\text{argmax}_{x \in X_i} f_i(x)$ for each period i), one could instead use a dynamic definition of myopic decisions where in each period the firm picks the myopic single period optimal decision with the added constraint that the lower bound resulting from the decision in the previous period must also be satisfied. With this definition, $x_1, \ldots, x_m$ are myopic decisions if x_1 is in $\text{argmax}_{\{x : z_1 \leq x, x \in X_1\}} f_1(x)$ and x_i is in $\text{argmax}_{\{x : z_i(x_{i-1}) \leq x, x \in X_i\}} f_i(x)$ for $i = 2, \ldots, m$. With this definition, any myopic decisions x_i for periods $i = 1, \ldots, m$ must be feasible and hence would be optimal for the m period problem if x_i is in $\text{argmax}_{x \in X_i} f_i(x)$ for $i = 1, \ldots, m$.

(With the earlier definition of myopic decisions, a sequence of myopic decisions for each period may not be feasible for the m period problem even when some feasible myopic solution exists.) With this definition, the statement of Theorem 3.8.1 still holds.

Theorem 3.8.1 permits a firm to make a decision in the first period of an m period problem that is optimal for the m period problem even when information relevant for subsequent periods is not specifically known. Suppose that a firm knows z_1, X_1, and $f_1(x_1)$ and also knows that the general conditions of Theorem 3.8.1 having to do with periods $i = 1, \ldots m$ hold, but the firm does not know the actual specifications of $z_i(x_{i-1})$, X_i, and $f_i(x_i)$ for $i = 2, \ldots, m$. Then the firm has enough information to determine and carry out a myopic decision x'_1 for period 1 with $z_1 \leq x'_1$, and having carried out the decision x'_1 in period 1 there exist some decisions $x'_2, \ldots, x'_m$ for periods $i = 2, \ldots, m$ such that the decisions $x'_1, \ldots, x'_m$ are optimal for the m period problem.

Finding optimal decisions for all periods of an m period problem requires less computation when it is known that optimal myopic decisions exist, because then one need only solve m single-period problems instead of an m period dynamic programming problem.

3.8.2 Dynamic Selection Problem

Consider a dynamic version of the selection problem of Subsection 3.7.3. A firm operates in m successive periods. In each period, the firm selects some set of activities in which to engage. There is a finite set N_i of activities that may be undertaken in each period i. If an activity is available in a particular period then it is available is all subsequent periods, so N_i is a subset of N_{i+1} for $i = 1, \ldots, m-1$. Any particular activity can be selected at most once, so in each period i the firm can select any subset of activities from N_i that have not been selected in any previous period. Let X_i be the set of all activities selected in any of the first i periods and $X_0 = \emptyset$, so in each period i the firm can select any subset of $N_i \setminus X_{i-1}$ and $X_i \setminus X_{i-1}$ is the set of activities selected in that period. For each activity x in N_i, the firm receives an undiscounted net return $r_i(x)$ for selecting activity x in period i. The activities are related such that certain activities can be selected in some period only if certain other activities have also been selected no later than that period. This requirement is represented by a subset W of $N_m \times N_m$, where for (x', x'') in W the firm can select activity x' in period i (that is, x' is in $X_i \setminus X_{i-1}$) only if activity x'' is selected in one of the first i periods (that is, x'' is in X_i). One case of such a requirement would be where each activity is either a profitable project with $r_i(x)$ nonnegative or a costly facility with $r_i(x)$ nonpositive, each project can be performed at most once, performing a project in a period requires the use of a particular set of facilities at that time, and each facility can be used for projects in the period when the facility is acquired and in all subsequent periods. Assume that if x' is in N_i and (x', x'') is in W then x'' is in N_i. (Otherwise, it would never be possible to select activity x' in period i.) A subset X of N_i is a feasible cumulative selection for period i with respect to the requirements represented by W if and only if X is a closure in the network with nodes corresponding to the elements in N_m and a directed edge (x', x'') corresponding to each (x', x'') in W. Let $\mathscr{X}$ be the collection of all subsets X of N_m such that X is a feasible cumulative selection for period m with respect to the requirements of W. Then $\mathscr{X}$ is a sublattice of the power set of N_m (with the set inclusion ordering relation) because the collection of all closures is a sublattice as noted in Subsection 3.7.2. Let $\mathscr{X}_i$ be those elements of $\mathscr{X}$ that are subsets of N_i, so $\mathscr{X}_i$ is the collection of all feasible cumulative selections for period i. For $i = 1, \ldots, m$, $\mathscr{X}_i$ is a sublattice of the power set of N_i. There is a discount rate β in $(0, 1)$ such that, defining $\gamma = 1/(1 + \beta)$, each unit of undiscounted net profit in any period i has a value γ^{i-1} in period 1. The objective of the firm is to select feasible activities $X_i \setminus X_{i-1}$ in each period $i = 1, \ldots, m$ so as to maximize the total present value of its net profit. The problem of the firm, the **dynamic selection problem**, is to

$$\text{maximize } \textstyle\sum_{i=1}^{m} \gamma^{i-1} \sum_{x \in X_i \setminus X_{i-1}} r_i(x) \tag{3.8.1}$$

$$\text{subject to } X_i \text{ in } \mathscr{X}_i \text{ for } i = 1, \ldots, m,$$

$$X_{i-1} \subseteq X_i \text{ for } i = 2, \ldots, m, \text{ and } X_0 = \emptyset.$$

Now the task is to transform the firm's objective function in (3.8.1) into a form for which Theorem 3.8.1 can be applied. (The present formulation already anticipates the model of Subsection 3.8.1 by taking the decision variables as the cumulative selections $X_1, \ldots, X_m$ up to and including each period rather than the selections $X_1 \setminus X_0, \ldots, X_m \setminus X_{m-1}$ in each period.) Define

$$r_i'(x) = r_i(x) - \gamma r_{i+1}(x) \text{ for each } x \text{ in } N_i \text{ and } i = 1, \ldots m - 1,$$

and

$$r_m'(x) = (1 - \gamma) r_m(x) \text{ for each } x \text{ in } N_m.$$

The return $r_i'(x)$ for $i < m$ takes the actual return from selecting activity x in period i and reduces it by the discounted value of the return from selecting the same activity in the next period (that is, by the loss from not being able to select that activity in the next period). Let

$$f_i(X_i) = \textstyle\sum_{x \in X_i} r_i'(x) \text{ for } X_i \subseteq N_i \text{ and } i = 1, \ldots m.$$

The function $f_i(X_i)$ is the net profit for choosing a (single period) selection of activities X_i from N_i for a selection problem with a net return $r_i'(x)$ from selecting any activity x in N_i. The firm's objective function in (3.8.1) can be transformed into the present value of the new net profit functions for the m single period selection problems since

$$\begin{aligned}
&\sum_{i=1}^{m} \gamma^{i-1} \sum_{x\in X_i\setminus X_{i-1}} r_i(x) \\
&\quad = \sum_{i=1}^{m-1}\Big(\gamma^{i-1}\sum_{x\in X_i} r_i(x) - \gamma^{i}\sum_{x\in X_i} r_{i+1}(x)\Big) + \gamma^{m-1}\sum_{x\in X_m} r_m(x) \\
&\quad = \sum_{i=1}^{m-1} \gamma^{i-1}\sum_{x\in X_i} r_i'(x) + (\gamma^{m-1}/(1-\gamma))\sum_{x\in X_m} r_m'(x) \\
&\quad = \sum_{i=1}^{m-1} \gamma^{i-1} f_i(X_i) + (\gamma^{m-1}/(1-\gamma)) f_m(X_m).
\end{aligned}$$

The dynamic selection problem thus becomes

$$\begin{gathered}
\text{maximize } \sum_{i=1}^{m-1} \gamma^{i-1} f_i(X_i) + (\gamma^{m-1}/(1-\gamma)) f_m(X_m) \\
\text{subject to } X_i \text{ in } \mathcal{X}_i \text{ for } i = 1,\ldots,m, \\
X_{i-1} \subseteq X_i \text{ for } i = 2,\ldots,m, \text{ and } X_0 = \emptyset.
\end{gathered}$$

Using the observation of part (d) of Example 2.2.1, the domain $\mathcal{X}_i$ of each single period net profit function $f_i(X_i)$ can be taken as a sublattice of $R^{|N_i|}$. Therefore, the present formulation of the dynamic selection problem fits the form considered in Subsection 3.8.1. Theorem 3.8.2 gives conditions for the existence of optimal myopic decisions. A real-valued function $g(y)$ on a set Y of consecutive integers is **concave** if $g(y+1) - g(y)$ is decreasing in y on Y.

Theorem 3.8.2. *If $r_i(x)$ is increasing and concave in i for each activity x, then optimal myopic decisions exist for the firm. In particular, a greatest myopic selection decision (that is, the greatest element of* $\text{argmax}_{X\in\mathcal{X}_i} f_i(X)$*) exists for each period i, and these myopic selection decisions are optimal.*

Proof. Because $\mathcal{X}_i$ is the intersection of the power set $\mathcal{P}(N_i)$ with the collection of all closures in N_m, $\mathcal{P}(N_i)$ is increasing in i (since $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_m$), and the collection of all closures in N_m is a sublattice, $\mathcal{X}_i$ is increasing in i by Theorem 2.4.2. By Theorem 3.8.1, it now suffices to show that $f_i(X)$ is supermodular in (X, i) on $\{(X, i) : X \in \mathcal{X}_i \text{ and } i = 1,\ldots,m\}$. Pick any (X', i') and (X'', i'') with X' in $\mathcal{X}_{i'}$ and X'' in $\mathcal{X}_{i''}$. Without loss of generality, suppose that $i' \le i''$. Then

$$\begin{aligned}
&f_{i''}(X'\cup X'') + f_{i'}(X'\cap X'') - f_{i'}(X') - f_{i''}(X'') \\
&\quad = \sum_{x\in X'\cup X''} r_{i''}'(x) + \sum_{x\in X'\cap X''} r_{i'}'(x) \\
&\qquad - \sum_{x\in X'} r_{i'}'(x) - \sum_{x\in X''} r_{i''}'(x) \\
&\quad = \sum_{x\in X'\setminus X''} r_{i''}'(x) - \sum_{x\in X'\setminus X''} r_{i'}'(x).
\end{aligned} \tag{3.8.2}$$

If $i'' < m$,

$$\begin{aligned}
&\textstyle\sum_{x\in X'\setminus X''} r'_{i''}(x) - \sum_{x\in X'\setminus X''} r'_{i'}(x)\\
&\quad = \textstyle\sum_{x\in X'\setminus X''}(r_{i''}(x) - \gamma r_{i''+1}(x) - r_{i'}(x) + \gamma r_{i'+1}(x))\\
&\quad = \textstyle\sum_{x\in X'\setminus X''}((1-\gamma)(r_{i''}(x) - r_{i'}(x)) \qquad (3.8.3)\\
&\qquad\qquad + \gamma(r_{i''}(x) - r_{i''+1}(x) - r_{i'}(x) + r_{i'+1}(x)))\\
&\quad \geq 0
\end{aligned}$$

because $r_i(x)$ is increasing and concave in i for each x.

If $i' < i'' = m$,

$$\begin{aligned}
&\textstyle\sum_{x\in X'\setminus X''} r'_{i''}(x) - \sum_{x\in X'\setminus X''} r'_{i'}(x)\\
&\quad = \textstyle\sum_{x\in X'\setminus X''}((1-\gamma)r_m(x) - r_{i'}(x) + \gamma r_{i'+1}(x))\\
&\quad = \textstyle\sum_{x\in X'\setminus X''}((1-\gamma)(r_m(x) - r_{i'}(x)) + \gamma(r_{i'+1}(x) - r_{i'}(x)) \qquad (3.8.4)\\
&\quad \geq 0
\end{aligned}$$

because $r_i(x)$ is increasing in i for each x.

The proof is complete by (3.8.2), (3.8.3), (3.8.4), and the observation that the expression in (3.8.2) equals 0 if $i' = i''$. □

Because each single period problem of maximizing $f_i(X_i)$ over X_i in $\mathscr{X}_i$ is a maximum closure problem, one can solve the dynamic selection problem under the hypotheses of Theorem 3.8.2 by solving m minimum cut problems as in Subsection 3.7.2.

3.9 Markov Decision Processes and Property-Inducing Stochastic Transformations

Subsection 3.9.2 studies Markov decision processes with the state in each period considered as a parameter. Conditions are given for optimal decisions to increase with the state in each period and for the components of the parameter to be complementary. Examples involve advertising, pricing, and the maintenance of an unreliable system. Subsection 3.9.1 develops properties of certain property-inducing stochastic transformations, which are used in the analysis of Subsection 3.9.2. This section is based on material in Topkis [1968]. Heyman and Sobel [1984] consider the present conditions for increasing optimal solutions in Markov decision processes and exhibit a number of examples. Serfozo [1976] gives other conditions, also with a supermodularity property, for increasing optimal solutions in Markov decision processes. White [1980] develops related conditions for increasing optimal solutions in partially observed Markov decision processes. Glasserman and Yao [1994], Heyman and Sobel [1984], and references therein give further theory and applications having to do with increasing optimal solutions and the role of supermodularity in stochastic optimization problems.

Assume throughout this section that, when needed, all sets are Borel sets and all real-valued functions are Borel measurable and integrable with respect to appropriate distribution functions.

3.9.1 Property-Inducing Stochastic Transformations

This subsection considers conditions under which taking the expected value of any increasing function with respect to each distribution function in a parameterized collection of distribution functions induces certain properties on the resulting function of the parameter. Lehmann [1955] gives such conditions where the property induced is that the function is increasing (or decreasing). Similar conditions are given here where all functions having the property form a closed convex cone. Supermodularity (or submodularity) and convexity (or concavity) are additional properties satisfying this general condition.

Let $\{F_t(w) : t \in T\}$ be a collection of distribution functions on R^n that are indexed by a parameter t, with t contained in a subset T of R^m. For a subset S of R^n and t in T, let $\int_S dF_t(w)$ be the probability measure of S with respect to the distribution function $F_t(w)$. For a real-valued function $h(w)$ on R^n and t in T, $\int h(w)dF_t(w)$ is the expected value of $h(w)$ with respect to $F_t(w)$. If $\int_S dF_t(w)$ is an increasing (decreasing) function of t on T for each increasing set S in R^n, then $F_t(w)$ is **stochastically increasing** (**stochastically decreasing**) in t on T. If $T = \{1, 2\}$ and $F_t(w)$ is stochastically increasing in t on T, then $F_2(w)$ is **stochastically larger** than $F_1(w)$. If T is a sublattice of R^m and $\int_S dF_t(w)$ is a supermodular (submodular) function of t on T for each increasing set S in R^n, then $F_t(w)$ is **stochastically supermodular** (**stochastically submodular**) in t on T. If T is a convex set and $\int_S dF_t(w)$ is a convex (concave) function of t on T for each increasing set S in R^n, then $F_t(w)$ is **stochastically convex** (**stochastically concave**) in t on T.

Lehmann [1955] establishes part (a) of Corollary 3.9.1, giving necessary and sufficient conditions for a collection of distribution functions to be stochastically increasing (decreasing). Theorem 3.9.1 generalizes this result from the increasing property (for the expectation) to any property where the set of all functions having the property is a closed convex cone. Athey [1995] gives a more general characterization result, where the particular limitation of the Theorem 3.9.1 characterization that the integrated function $h(w)$ be increasing is extended to let $h(w)$ be in any closed convex cone, and further shows that there is no stronger characterization than that developed in Athey [1995]. Corollary 3.9.1 follows as special cases of Theorem 3.9.1, yielding Lehmann's [1955] result as well as analogous results for stochastic supermodularity (submodularity) and stochastic convexity (concavity). Part (b) of Corollary 3.9.1 further relies on Lemma 2.6.1.

Theorem 3.9.1. *Suppose that T is a subset of R^m, $\{F_t(w) : t \in T\}$ is a collection of distribution functions on R^n, and $\mathcal{F}$ is a closed (in the topology of pointwise convergence) convex cone of real-valued functions on T. Then $\int_S dF_t(w)$ is in $\mathcal{F}$ for each increasing set S in R^n if and only if $\int h(w)dF_t(w)$ is in $\mathcal{F}$ for each increasing real-valued function $h(w)$ on R^n.*

Proof. Let $I(w;S)$ be the indicator function of any subset S of R^n, so $I(w;S)$ is defined on $R^n \times \mathcal{P}(R^n)$ such that $I(w;S) = 1$ if w is in S and $I(w;S) = 0$ if w is not in S.

Sufficiency follows because $\int_S dF_t(w) = \int I(w;S)dF_t(w)$ and the indicator function $I(w;S)$ of any increasing set S is increasing in w.

Now suppose that $\int_S dF_t(w)$ is in $\mathcal{F}$ for each increasing set S in R^n. It suffices to prove the necessity part of the result for $h(w)$ nonnegative, because the proof for nonpositive $h(w)$ is similar and one can express an arbitrary function $h(w)$ as the sum of a nonnegative function and a nonpositive function as $h(w) = h(w) \vee 0 + h(w) \wedge 0$. Pick any nonnegative increasing function $h(w)$. For $k = 1, 2, \ldots$, let $i(k) = k2^k + 1$. For $i = 1, \ldots, i(k) - 1$ and $k = 1, 2, \ldots$, let $S(i,k) = \{w : (i-1)/2^k \le h(w) < i/2^k\}$ and $a_{i,k} = (i-1)/2^k$. Let $S(i(k),k) = \{w : (i(k)-1)/2^k \le h(w)\}$ and $a_{i(k),k} = (i(k)-1)/2^k$. Define $h_k(w) = \sum_{i=1}^{i(k)} a_{i,k} I(w; S_{i,k})$ for w in R^n and $k = 1, 2, \ldots$. The functions $\{h_k(w) : k = 1, 2, \ldots \}$ converge upwards to $h(w)$. Then $\int h_k(w)dF_t(w) = \sum_{i=1}^{i(k)} a_{i,k} \int I(w; S_{i,k})dF_t(w) = \sum_{i=2}^{i(k)} a_{i,k} \int I(w; S_{i,k})dF_t(w) = \sum_{i=2}^{i(k)} (a_{i,k} - a_{i-1,k})(\sum_{j=i}^{i(k)} \int I(w; S_{j,k})dF_t(w)) = \sum_{i=2}^{i(k)} (a_{i,k} - a_{i-1,k}) \int I(w; \cup_{j=i}^{i(k)} S_{j,k})dF_t(w)$ is in $\mathcal{F}$ for each k because $\int I(w; \cup_{j=i}^{i(k)} S_{j,k})dF_t(w) = \int_{\cup_{j=i}^{i(k)} S_{j,k}} dF_t(w)$ is in $\mathcal{F}$ for all i and k by hypothesis (since $\cup_{j=i}^{i(k)} S_{j,k}$ is an increasing set), $a_{i,k}$ is increasing in i for each k, and $\mathcal{F}$ is a convex cone. By the monotone convergence theorem (Billingsley [1979]), $\int h(w)dF_t(w) = \lim_{k\to\infty} \int h_k(w)dF_t(w)$ for each t in T and so $\int h(w)dF_t(w)$ is in $\mathcal{F}$ because $\mathcal{F}$ is closed in the topology of pointwise convergence. □

Corollary 3.9.1. *Suppose that T is a subset of R^m and $\{F_t(w) : t \in T\}$ is a collection of distribution functions on R^n.*

(a) *$F_t(w)$ is stochastically increasing (decreasing) in t on T if and only if $\int h(w)dF_t(w)$ is increasing (decreasing) in t on T for each increasing real-valued function $h(w)$ on R^n.*
(b) *If T is a sublattice of R^m, then $F_t(w)$ is stochastically supermodular (submodular) in t on T if and only if $\int h(w)dF_t(w)$ is supermodular (submodular) in t on T for each increasing real-valued function $h(w)$ on R^n.*
(c) *If T is a convex subset of R^m, then $F_t(w)$ is stochastically convex (concave) in t on T if and only if $\int h(w)dF_t(w)$ is convex (concave) in t on T for each increasing real-valued function $h(w)$ on R^n.*

It is well-known that a collection of distribution functions $\{F_t(w) : t \in T\}$ on R^1 is stochastically increasing in t on a subset T of R^m if and only if $1 - F_t(w)$ is increasing in t on T for each w in R^1. This result, restated in part (b) of Lemma 3.9.1, is generalized in part (a) of Lemma 3.9.1 and has analogs for stochastic supermodularity and stochastic convexity as given in part (c) and part (d) of Lemma 3.9.1.

Lemma 3.9.1. *Suppose that T is a subset of R^m and $\{F_t(w) : t \in T\}$ is a collection of distribution functions on R^1.*
(a) *If $\mathcal{F}$ is a closed (in the topology of pointwise convergence) subset of the collection of all real-valued functions on T, then $\int_S dF_t(w)$ is in $\mathcal{F}$ for each increasing set S in R^1 if and only if $1 - F_t(w)$ is in $\mathcal{F}$ for each w in R^1.*
(b) *The collection of distribution functions $F_t(w)$ is stochastically increasing (decreasing) in t on T if and only if $1 - F_t(w)$ is increasing (decreasing) in t on T for each w in R^1.*
(c) *If T is a sublattice of R^m, then $F_t(w)$ is stochastically supermodular (submodular) in t on T if and only if $1 - F_t(w)$ is supermodular (submodular) in t on T for each w in R^1.*
(d) *If T is convex, then $F_t(w)$ is stochastically convex (concave) in t on T if and only if $1 - F_t(w)$ is convex (concave) in t on T for each w in R^1.*

Proof. A proof is given for part (a). Part (b), part (c), and part (d) follow directly from part (a).

If $\int_S dF_t(w)$ is in $\mathcal{F}$ for each increasing set S in R^1 and w' is in R^1, then $1 - F_t(w') = \int_{(w',\infty)} dF_t(w)$ is in $\mathcal{F}$ because (w', ∞) is an increasing set in R^n.

Suppose that $1 - F_t(w)$ is in $\mathcal{F}$ for each w in R^1. Each increasing set in R^1, other than R^1 itself and the empty set, is of the form (w, ∞) or $[w, \infty)$ for some w in R^1. For w' in R^1, $\int_{(w',\infty)} dF_t(w) = 1 - F_t(w')$ is in $\mathcal{F}$ by hypothesis. For w' in R^1, $\int_{[w',\infty)} dF_t(w) = \lim_{z \to w', z < w'} \int_{(z,\infty)} dF_t(w) = \lim_{z \to w', z < w'} (1 - F_t(z))$, which is in $\mathcal{F}$ by hypothesis and because $\mathcal{F}$ is closed. □

Example 3.9.1 gives some common families of distribution functions on R^1 that, by part (b) and part (d) of Lemma 3.9.1, are stochastically decreasing and stochastically convex.

Example 3.9.1. Each of the following collections of distribution functions on R^1 is stochastically decreasing and stochastically convex in the parameter t.
(a) For t in $T = (0, \infty)$ and any fixed $\alpha > 0$, the **Weibull distribution** has the distribution function $F_t(w) = 0$ if $w \le 0$ and $F_t(w) = 1 - e^{-tw^\alpha}$ if $w \ge 0$. The special case for $\alpha = 1$ is the **exponential distribution**.
(b) For t in $T = (0, \infty)$ and any fixed real α, the **uniform distribution** has the distribution function $F_t(w) = 0$ if $w \le \alpha$, $F_t(w) = t(w - \alpha)$ if $\alpha \le w \le \alpha + 1/t$, and $F_t(w) = 1$ if $\alpha + 1/t \le w$.
(c) For t in $T = [0, 1]$, the **Bernoulli distribution** has the distribution function $F_t(w) = 0$ if $w < 0$, $F_t(w) = t$ if $0 \le w < 1$, and $F_t(w) = 1$ if $1 \le w$.

While it is well-known that a random vector composed of two independent stochastically increasing random vectors is itself stochastically increasing (Shaked and Shanthikumar [1994]), the analogous result is not true for stochastically convex random vectors. As the following example shows, the sum of two independent, identically distributed, stochastically convex, real random variables need not be stochastically convex.

Example 3.9.2. Let $G_t(w)$ be the distribution function for the sum of two (stochastically decreasing and stochastically convex) independent Bernoulli random variables, each with the Bernoulli distribution function $F_t(w)$ as in part (c) of Example 3.9.1. For t in $T = [0, 1]$, $G_t(w) = 0$ if $w < 0$, $G_t(w) = t^2$ if $0 \le w < 1$, $G_t(w) = 2t - t^2$ if $1 \le w < 2$, and $G_t(w) = 1$ if $2 \le w$. By part (d) of Lemma 3.9.1, this is not stochastically convex because $1 - G_t(w) = 1 - t^2$ for $0 \le w < 1$ is not convex in t on $[0, 1]$.

Lemma 3.9.2 shows that a stochastically decreasing and stochastically convex collection of distribution functions (as in each part of Example 3.9.1) can be used to generate a stochastically supermodular collection of distribution functions.

Lemma 3.9.2. *Suppose that T is a convex subset of R^1, $\{F_t(w) : t \in T\}$ is a collection of distribution functions on R^n that is stochastically decreasing and stochastically convex in t on T, Z is a sublattice of R^m, and $t(z)$ is an increasing submodular function from Z into T. Then $F_{t(z)}(w)$ is stochastically supermodular in z on Z.*

Proof. Let S be any increasing set in R^n. Because $F_t(w)$ is stochastically decreasing and stochastically convex, $\int_S dF_t(w)$ is decreasing and convex in t.

Then $\int_S dF_{t(z)}(w)$ is supermodular in z on Z by Lemma 2.6.4 because a decreasing convex function of an increasing submodular function is supermodular. Therefore, $F_{t(z)}(w)$ is stochastically supermodular in z on Z. □

Lemma 3.9.3 identifies a collection of stochastically supermodular distribution functions that is useful for the maintenance application in Example 3.9.5.

Lemma 3.9.3. *If Z_1 and Z_2 are subsets of R^1, $Z_1 \sqsubseteq Z_2$, $F_{z,1}(w)$ is a distribution function on R^n for each z in Z_1 such that $F_{z,1}(w)$ does not depend on z, and $F_{z,2}(w)$ is a distribution function on R^n for each z in Z_2 such that $F_{z,2}(w)$ is stochastically increasing in z on Z_2, then $F_{z,i}(w)$ is stochastically supermodular in (z, i) on $(Z_1 \times \{1\}) \cup (Z_2 \times \{2\})$.*

Proof. Let S be any increasing set in R^n. Because $Z_1 \sqsubseteq Z_2$, $(Z_1 \times \{1\}) \cup (Z_2 \times \{2\})$ is a sublattice of R^2 by part (b) of Theorem 2.4.5. Pick any (z', i') and (z'', i'') in $(Z_1 \times \{1\}) \cup (Z_2 \times \{2\})$. Without loss of generality, suppose that $i' \leq i''$.

If $i' = i''$, then

$$\int_S dF_{(z',i')\vee(z'',i'')}(w) + \int_S dF_{(z',i')\wedge(z'',i'')}(w) - \int_S dF_{z',i'}(w) - \int_S dF_{z'',i''}(w)$$
$$= \int_S dF_{z'\vee z'',i'}(w) + \int_S dF_{z'\wedge z'',i'}(w) - \int_S dF_{z',i'}(w) - \int_S dF_{z'',i'}(w) = 0$$

because Z_1 and Z_2 being subsets of R^1 implies that $\{z', z''\} = \{z' \vee z'', z' \wedge z''\}$.

If $i' < i''$, then $1 = i' < i'' = 2$ and

$$\int_S dF_{(z',i')\vee(z'',i'')}(w) + \int_S dF_{(z',i')\wedge(z'',i'')}(w) - \int_S dF_{z',i'}(w) - \int_S dF_{z'',i''}(w)$$
$$= \int_S dF_{z'\vee z'',2}(w) + \int_S dF_{z'\wedge z'',1}(w) - \int_S dF_{z',1}(w) - \int_S dF_{z'',2}(w)$$
$$= \int_S dF_{z'\vee z'',2}(w) - \int_S dF_{z'',2}(w) \geq 0,$$

where the second equality follows because $F_{z,1}(w)$ does not depend on z and the inequality follows because $F_{z,2}(w)$ is stochastically increasing in z. □

3.9.2 Markov Decision Processes

Consider a nonstationary discounted finite horizon discrete time Markov decision process with periods $i = 1, \ldots, k$. Let t and x denote the state and decision, respectively, in a given period. The collection of all possible states in period i is a subset T_i of R^m. In state t and period i the decision x is restricted to a finite nonempty subset $X_{t,i}$ of R^n. Let $S_i = \{(x, t) : t \in T_i,$

$x \in X_{t,i}\}$. The (expected) net return resulting from being in state t and making decision x in period i is $r_i(x, t)$, where $r_i(x, t)$ is bounded on S_i for each i. There is a discount rate β in $[0, 1]$ such that, defining $\gamma = 1/(1+\beta)$, each unit of undiscounted net return in any period i has a value γ^{i-1} in period 1. The distribution function for the state w in period $i+1$ given the state t and decision x in period i is $F_{x,t,i}(w)$. The objective is to maximize the total expected present value of all returns.

Let $f_i(t)$ be the maximum over all decisions for all states in periods $i, \ldots, k$ of the present value in period i of all expected returns in periods $i, \ldots, k$, given that the state in period i is t. Let $g_i(x, t)$ be the maximum over all decisions for all states in periods $i+1, \ldots, k$ of the present value in period i of all expected returns in periods $i, \ldots, k$, given that the state in period i is t and the decision in period i is x in $X_{t,i}$. Starting with $g_k(x, t) = r_k(x, t)$, one can compute $f_i(t)$ and $g_i(x, t)$ recursively for each i from

$$f_i(t) = \max\{g_i(x, t) : x \in X_{t,i}\} \quad \text{for } 1 \le i \le k, \tag{3.9.1}$$

and

$$g_i(x, t) = r_i(x, t) + \gamma \textstyle\int f_{i+1}(w) dF_{x,t,i}(w) \quad \text{for } 1 \le i \le k-1. \tag{3.9.2}$$

Because each $r_i(x, t)$ is bounded, each $g_i(x, t)$ and $f_i(t)$ is bounded. Because each $X_{t,i}$ is finite and nonempty, the maximum in (3.9.1) is always attained and each set of optimal decisions $\operatorname{argmax}_{x \in X_{t,i}} g_i(x, t)$ is nonempty.

Lemma 3.9.4 gives conditions for $f_i(t)$ to be an increasing function of the state t for each period i.

Lemma 3.9.4. *If $X_{t',i}$ is a subset of $X_{t'',i}$ for each period i and all states t' and t'' in T_i with $t' \le t''$, $r_i(x, t)$ is increasing in t on the section of S_i at x for all x and i, and $F_{x,t,i}(w)$ is stochastically increasing in t on the section of S_i at x for all x and i, then $f_i(t)$ is increasing in t on T_i for each i.*

Proof. If $g_i(x, t)$ is increasing in t for each x, then $f_i(t)$ is increasing in t by (3.9.1) and the hypothesis that $X_{t',i}$ is a subset of $X_{t'',i}$ for all t' and t'' in T_i with $t' \le t''$. Hence, it suffices to show that $g_i(x, t)$ is increasing in t for all x and i. Pick any i with $1 \le i < k$, and suppose that $g_{i+1}(x, t)$ is increasing in t given x. This statement is true for $i = k-1$, so for the proof to follow by induction it suffices to show that $g_i(x, t)$ is increasing in t. Pick t' and t'' in T_i with $t' \le t''$ and pick any x' in $X_{t',i}$. Then

$$\begin{aligned} g_i(x', t') &= r_i(x', t') + \gamma \textstyle\int f_{i+1}(w) dF_{x',t',i}(w) \\ &\le r_i(x', t'') + \gamma \textstyle\int f_{i+1}(w) dF_{x',t'',i}(w) = g_i(x', t''), \end{aligned}$$

where the inequality follows by part (a) of Corollary 3.9.1 and because $X_{t',i}$ is a subset of $X_{t'',i}$, $r_i(x', t)$ is increasing in t on the section of S_i at x', and $F_{x',t,i}(w)$ is stochastically increasing in t on the section of S_i at x'. Thus $g_i(x, t)$ is increasing in t. □

Theorem 3.9.2 uses Lemma 3.9.4 to establish conditions under which, in each period, the components of the state are complementary (with respect to the optimal return function $f_i(t)$ by Theorem 2.6.1) and optimal decisions increase with the state. The proof for Theorem 3.9.2 involves using Theorem 2.7.6 (the preservation of supermodularity under the maximization operation) to establish the supermodularity of the optimal return function with this property being maintained from period to period and then using Theorem 2.8.2 (increasing optimal solutions) to establish that optimal decisions increase with the state. An analogous approach is used for corresponding results in different dynamic programming models (Amir [1996]; Amir, Mirman, and Perkins [1991]; and Subsection 3.10.2).

Theorem 3.9.2. *Suppose that S_i is a sublattice of R^{n+m} for each i, $X_{t',i}$ is a subset of $X_{t'',i}$ for each period i and all states t' and t'' in T_i with $t' \leq t''$, $r_i(x, t)$ is increasing in t on the section of S_i at x for all x and i and is supermodular in (x, t) on S_i for each i, and $F_{x,t,i}(w)$ is stochastically increasing in t on the section of S_i at x for all x and i and is stochastically supermodular in (x, t) on S_i for each i. For each period i,*

(a) *$g_i(x, t)$ is supermodular in (x, t) on S_i;*

(b) *$f_i(t)$ is supermodular in t on T_i;*

(c) *the set of optimal decisions* $\operatorname{argmax}_{x \in X_{t,i}} g_i(x, t)$ *is increasing in the state t on T_i; and*

(d) *there is a greatest (least) optimal decision for each state t and this greatest (least) optimal decision is increasing in t.*

Proof. Part (a) holds for $i = k$ by hypothesis. Pick any i with $1 \leq i < k$. By Lemma 3.9.4, $f_{i+1}(t)$ is increasing in t on T_{i+1}. Because S_i is a sublattice of R^{n+m} and $F_{x,t,i}(w)$ is stochastically supermodular in (x, t) on S_i, $\int f_{i+1}(w) dF_{x,t,i}(w)$ is supermodular in (x, t) on S_i by part (b) of Corollary 3.9.1. Then part (a) holds by (3.9.2) and part (a) and part (b) of Lemma 2.6.1 and because $r_i(x, t)$ is supermodular in (x, t). Part (b) follows from part (a), (3.9.1), and Theorem 2.7.6. Part (c) follows from part (a) and Theorem 2.8.2. Part (d) follows from part (a) and Theorem 2.8.3. □

Now consider the infinite period stationary discounted version of this Markov decision process with the additional assumptions that γ is in $(0, 1)$ and that each constraint set $X_{t,i}$ does not depend on the state t. Drop the subscript t from $X_{t,i}$. Drop the subscript i indicating dependency on the period i where stationarity makes it no longer necessary. Denote the dependence of $f_i(t)$ on the total number of periods k as $f_{i,k}(t)$. Equation (3.9.1) and equation (3.9.2) together with $f_{i,k}(t) = f_{i+1,k+1}(t)$ yield

$$f_{1,k}(t) = \max\{r(x,t) + \gamma \int f_{1,k-1}(w) dF_{x,t}(w) : x \in X\}. \tag{3.9.3}$$

Because $r(x,t)$ is bounded on S, $f_{1,k}(t)$ converges uniformly on T as k goes to ∞ to some real-valued function $f(t)$. By (3.9.3), $f(t)$ satisfies the optimality equation

$$f(t) = \max\{r(x,t) + \gamma \int f(w) dF_{x,t}(w) : x \in X\} \tag{3.9.4}$$

for t in T. Because $f(t)$ satisfies the optimality equation (3.9.4), it follows from a result of Blackwell [1965] that $f(t)$ is the maximum present value in period 1 of all expected returns from beginning in state t in period 1. If the hypotheses of Theorem 3.9.2 hold, then each $f_{1,k}(t)$ is increasing and supermodular on T by Lemma 3.9.4 and part (b) of Theorem 3.9.2 and so $f(t)$ is increasing and supermodular on S by part (c) of Lemma 2.6.1. Let

$$g(x,t) = r(x,t) + \gamma \int f(w) dF_{x,t}(w) \tag{3.9.5}$$

for (x,t) in S. The set of optimal decisions $\text{argmax}_{x \in X}\, g(x,t)$ in each period given that one is in state t can be determined from (3.9.4) if $f(t)$ is known. If the hypotheses of Theorem 3.9.2 hold, then, as with the proof of Theorem 3.9.2, the set of optimal decisions $\text{argmax}_{x \in X}\, g(x,t)$ is increasing in the state t on T and greatest and least optimal decisions exist for each t and are increasing in t.

The results of this subsection generalize to the case in which any possible state in any period consists of a pair (t_1, t_2), where the first component t_1 is generated through time as a Markov process. Then t_1 may be thought of as representing an uncontrollable random environment and t_2 would represent that portion of the state vector that may be partially controlled by the choice of decisions. Versions of Lemma 3.9.4 and Theorem 3.9.2 hold if the assumptions are restated for each fixed t_1 with only t_2 and x allowed to vary and if the stochastically increasing and stochastically supermodular assumptions are stated in terms of conditional distributions of t_2 given t_1. Here, the set of optimal decisions $\text{argmax}_{x \in X}\, g(x,t)$ is increasing in t_2 for each fixed t_1. Jorgenson, McCall, and Radner [1967] include such an environmental variable in a maintenance model for which they establish monotone optimal solutions.

Following are three examples of Markov decision processes for which the monotonicity result of Theorem 3.9.2 or of the infinite horizon discounted model discussed thereafter is applicable. These examples consider optimal advertising (Example 3.9.3), optimal pricing (Example 3.9.4), and optimal maintenance of an unreliable system (Example 3.9.5). Example 3.9.5 generalizes an example of Jorgenson, McCall, and Radner [1967]. In these examples, assume without further mention the general structure of the above decision model together with the appropriate regularity conditions. The emphasis is on the characteristics of an individual period and establishing the conditions of Theorem 3.9.2. These examples use the above notation except that the subscript i is omitted.

Example 3.9.3. Consider a firm that markets a single product and chooses its advertising budget in each period based on the level of sales in the previous period. After observing the sales t in the last period, the firm decides on its advertising budget $-x$ for the present period where x (the negative of the advertising budget) is taken from a nonempty finite set of possibilities X. If the sales level in the current period is w, then the net return (total revenue minus total cost) exclusive of advertising expenditures from such sales is $v(w)$. Assume that higher sales are always more desirable, so $v(w)$ is increasing in w. Given sales t in the preceding period and an advertising budget $-x$ in the present period, the distribution function for sales in the present period is $F_{x,t}(w)$ and the expected net return in the present period is $r(x, t) = x + \int v(w)dF_{x,t}(w)$. Suppose that $F_{x,t}(w)$ is stochastically increasing in t and stochastically supermodular in (x, t). This means that higher sales in one period increase the probability of higher sales in the following period and that a marginal amount of additional advertising expenditure is more effective the lower sales were in the preceding period. The assumption that $F_{x,t}(w)$ is stochastically increasing in t may reflect consumer loyalty and habit. The assumption that $F_{x,t}(w)$ is stochastically supermodular in (x, t) may reflect there being more potential new customers to be influenced by additional advertising for lower prior sales. Because $F_{x,t}(w)$ is stochastically increasing in t and stochastically supermodular in (x, t) and $v(w)$ is increasing in w, part (a) and part (b) of Corollary 3.9.1 imply that $r(x, t) = x + \int v(w)dF_{x,t}(w)$ is increasing in t and supermodular in (x, t). Thus, the assumptions of Theorem 3.9.2 hold and optimal advertising expenditures in each period decrease with the sales in the preceding period. As sales decrease, it is optimal to put greater effort into building them back up through advertising.

Example 3.9.4. Consider a firm that markets a single product and chooses its price x in each period based on the level of sales t in the previous period. The price x must be chosen from a nonempty finite set X. There is a unit production cost c. The product is never priced below marginal cost, so $x \geq c$ for each x in X. There is an upper bound u on production, and in each period the firm satisfies as much of that period's demand as possible with production in that period. Any demand that cannot be satisfied in the period in which it arrives is lost. No inventory is held from one period to the next. The distribution function for demand in any period is $F_{x,t}(w)$. Then the expected net return in any period is $r(x, t) = (x - c) \int (u \wedge w) dF_{x,t}(w)$. Assume that $F_{x,t}(w)$ is stochastically increasing in t and stochastically supermodular in (x, t), with the rationale for these assumptions being as in Example 3.9.3. Because $F_{x,t}(w)$ is stochastically increasing in t and $(x - c)(u \wedge w)$ is increasing in w, $r(x, t)$ is increasing in t by part (a) of Corollary 3.9.1. The return function $r(x, t)$ is supermodular in (x, t) because, for sales $t' \leq t''$ and prices $x' \leq x''$, the above hypotheses together with part (b) of Corollary 3.9.1 imply that

$$\begin{aligned}
&r(x'', t'') - r(x'', t') - r(x', t'') + r(x', t') \\
&\quad = (x'' - c)\big(\textstyle\int (u \wedge w) dF_{x'',t''}(w) - \int (u \wedge w) dF_{x'',t'}(w)\big) \\
&\qquad - (x' - c)\big(\textstyle\int (u \wedge w) dF_{x',t''}(w) - \int (u \wedge w) dF_{x',t'}(w)\big) \\
&\quad \geq (x'' - c)\big(\textstyle\int (u \wedge w) dF_{x'',t''}(w) - \int (u \wedge w) dF_{x'',t'}(w)\big) \\
&\qquad - (x' - c)\big(\textstyle\int (u \wedge w) dF_{x'',t''}(w) - \int (u \wedge w) dF_{x'',t'}(w)\big) \\
&\quad = (x'' - x')\big(\textstyle\int (u \wedge w) dF_{x'',t''}(w) - \int (u \wedge w) dF_{x'',t'}(w)\big) \geq 0.
\end{aligned}$$

Thus, the assumptions of Theorem 3.9.2 hold and optimal prices in each period increase with the sales in the preceding period.

Example 3.9.5. Consider a firm operating a system whose state is an m-vector t. A higher value of t indicates a more desirable state, as is made more precise below. The firm can partially control the system by choosing from among decisions $x = 1, \ldots, h$, where $h \geq 2$. Decision h is to do nothing to the system. Decision 1 is to completely replace the system. Decisions $2, \ldots, h - 1$ correspond to various overhaul operations on the system. Assume that the set of allowable decisions X_t in state t is increasing in t, and $X_{t'}$ is a subset of $X_{t''}$ for $t' \leq t''$. For instance, the firm may require that the system be either overhauled or replaced in very low states but that any decision is feasible in higher states. The replacement and overhaul operations are thorough enough that the distribution function $F_{x,t}(w)$ of the state in the next period does not depend on the present state t if the firm makes a decision x with $x \leq h - 1$. If the firm makes decision h and does nothing to the system, then a higher present state t makes it more likely that there is a higher state in the next period. Therefore, the distribution function $F_{h,t}(w)$ of the state in the next period is stochastically increasing in the present state t. If $m = 1$, then $F_{x,t}(w)$

is stochastically supermodular in (x, t) by Lemma 3.9.3 (but $F_{x,t}(w)$ may not be stochastically supermodular if $m > 1$). The firm receives the return $r(x, t)$ for making decision x in state t, and $r(x, t)$ is increasing in the state t. Assume that if $t' \leq t''$ then $r(x, t'')-r(x, t')$ is increasing in the decision x on $X_{t'}$. While it need not be true that $r(x, t)$ is supermodular or that $F_{x,t}(w)$ is stochastically supermodular if $m > 1$, one may use an alternative form of the hypotheses of Theorem 3.9.2 corresponding to Theorem 2.8.1 to confirm that the conclusion of part (c) of Theorem 3.9.2 still applies here. Thus optimal control decisions increase with the state of the system.

3.10 Stochastic Inventory Problems and Supermodularity-Preserving Stochastic Transformations

Subsection 3.10.2 considers multiperiod stochastic inventory problems with the cost functions and demand distributions depending on a parameter and gives conditions for optimal inventory decisions to increase with the parameter. Subsection 3.10.1 gives conditions under which certain parameterized stochastic transformations of certain functions preserve supermodularity, and these conditions are used in Subsection 3.10.2.

Assume throughout this section that, when needed, all real-valued functions are Borel measurable and integrable with respect to appropriate distribution functions.

3.10.1 Supermodularity-Preserving Stochastic Transformations

Theorem 3.10.1 provides general conditions for supermodularity to be preserved by parameterized stochastic transformations.

Theorem 3.10.1. *If X is a sublattice of R^n, T is a subset of R^1, $\{F_t(w) : t \in T\}$ is a stochastically increasing collection of distribution functions on R^m for t in T, and $f(x, t, w)$ is a real-valued function on $X \times T \times R^m$ that has increasing differences in (x, w) on $X \times R^m$ for fixed t in T and is supermodular in (x, t) on $X \times T$ for fixed w in R^m, then $g(x, t) = \int f(x, t, w)dF_t(w)$ is supermodular in (x, t) on $X \times T$.*

Proof. Pick any (x', t') and (x'', t'') in $X \times T$. Suppose, without loss of generality, that $t' \wedge t'' = t' \leq t'' = t' \vee t''$. Then

$$\begin{aligned} g(x', t') - g(x' \wedge x'', t') &= \int (f(x', t', w) - f(x' \wedge x'', t', w))dF_{t'}(w) \\ &\leq \int (f(x' \vee x'', t'', w) - f(x'', t'', w))dF_{t'}(w) \\ &\leq \int (f(x' \vee x'', t'', w) - f(x'', t'', w))dF_{t''}(w) \\ &= g(x' \vee x'', t'') - g(x'', t''), \end{aligned}$$

where the first inequality follows from the supermodularity of $f(x, t, w)$ in (x, t) together with Corollary 2.6.2 and the second inequality follows from part (a) of Corollary 3.9.1 together with the assumptions that $f(x, t, w)$ has increasing differences in (x, w) and that $F_t(w)$ is stochastically increasing in t. □

Corollary 3.10.1 follows as a special case of Theorem 3.10.1 by Theorem 2.6.1 and part (b) of Lemma 2.6.2. This result is used in the parametric inventory application of Subsection 3.10.2.

Corollary 3.10.1. *If T is a subset of R^1, $\{F_t(w) : t \in T\}$ is a stochastically increasing collection of distribution functions on R^1 for t in T, and $f(z, t)$ is supermodular in (z, t) on $R^1 \times T$ and is concave in z on R^1 for fixed t in T, then $g(x, t) = \int f(x - w, t) dF_t(w)$ is supermodular in (x, t) on $R^1 \times T$.*

Corollary 3.10.2 follows as a special case of Theorem 3.10.1 by Theorem 2.6.1 and part (a) of Lemma 2.6.2.

Corollary 3.10.2. *If T is a subset of R^1, $\{F_t(w) : t \in T\}$ is a stochastically increasing collection of distribution functions on R^n for t in T, and $f(z, t)$ is supermodular in (z, t) on $R^n \times T$ and is convex in z_i for $i = 1, \dots, n$, then $g(x, t) = \int f(x + w, t) dF_t(w)$ is supermodular in (x, t) on $R^n \times T$.*

3.10.2 Stochastic Inventory Problems

A firm produces n products in each of m successive time periods $i = 1, \dots, m$. A production decision in any time period is carried out without any time lag. Demand arrives in a period after the production decision is made for that period. All unsatisfied demand in one period is backlogged to the beginning of the next period. When demand is observed in a period, as much as possible of that demand and of any previously backlogged demand is immediately satisfied with inventory on hand including the production in that same period. A positive amount of demand for a product remains backlogged at the beginning or end of some period only if there is no inventory on hand for that product at that time. The net inventory level for each product is the difference between the stock on hand in inventory and the amount of backlogged demand. If z is the net inventory level for some product at the beginning or end of some period, then z nonnegative indicates z units of stock on hand and no backlogged demand for that product and z nonpositive indicates no stock on hand and $-z$ units of backlogged demand for that product. There are no bounds on the net inventory level. The only bound on the level of production is that it be nonnegative. (The analysis below can easily be modified to include an upper bound on production. Bounds on the net inventory level may be problematic only in that they could impact feasibility.) Consider a collection of different problems, where each problem, its costs, and its distribution functions are indexed with a parameter t contained in a parameter set T in R^1. The problem for a given t in T is **problem t**. The n-vector of demand is generated according to a known distribution function $F_{t,i}(w)$ in period i of problem t. The cost of producing a nonnegative n-vector y of products in period i of problem t is $c_i(y, t)$. If the n-vector of net inventory levels after production is z in period i of problem t, then there is a combined expected holding and penalty cost $L_i(z, t)$. There is a discount rate β in $[0, 1]$ such that, defining $\gamma = 1/(1+\beta)$, each unit of undiscounted cost in any period i is equivalent to a cost γ^{i-1} in period 1. Let $f_i(x, t)$ be the present value in period i of the expected costs in periods $i, \ldots, m$ of problem t, given that the n-vector of net inventory levels before production in period i is x and that the firm proceeds optimally in period i and thereafter. Assume that all minima are attained. Then $f_i(x, t)$ can be calculated recursively from

$$f_i(x, t) = \min_{\{z:z\geq x\}}(c_i(z-x, t) + L_i(z, t) + \gamma \int f_{i+1}(z-w, t)dF_{t,i}(w)) \tag{3.10.1}$$

$$\text{for } 1 \leq i \leq m$$

and

$$f_{m+1}(x, t) = 0.$$

Theorem 3.10.2, Theorem 3.10.3, and Theorem 3.10.4 give conditions for the optimal net inventory levels after production in each period to increase with the parameter t, where the production costs, the combined expected holding and penalty costs, and the demand distributions vary with t. These results extend Karlin [1960], who establishes the monotonicity of optimal inventory decisions with respect to variations in the distribution functions for demand in a single product problem. It also falls out of the present proofs that the optimal net inventory levels after production in each period increase with the net inventory level before production. Theorem 3.10.2 and Theorem 3.10.3 consider the case of a single product, with the former result having a convex production cost that does not depend on the parameter and the latter result having a linear production cost that may depend on the parameter. The standard convexity result stated in part (a) of Theorem 3.10.2 and part (a) of Theorem 3.10.3 facilitates the application of Corollary 3.10.1. Theorem 3.10.4 considers the case where there can be multiple products. The hypotheses of Theorem 3.10.3 and Theorem 3.10.4 are similar, except that the latter result further requires that the distribution functions for demand in each period be independent of the parameter and omits a convexity assumption on the combined expected holding and penalty cost function.

Theorem 3.10.2. *Suppose that there is a single product (so $m = 1$), the production cost $c_i(y, t)$ is convex in the production level y and is independent of the parameter t for each period i, the combined expected holding and penalty cost function $L_i(z, t)$ is convex in the inventory level z after production and is submodular in (z, t) for each period i, and the distribution function $F_{t,i}(w)$ for demand in period i of problem t is stochastically increasing in t on the parameter set T for each period i. For each i,*
(a) *$f_i(x, t)$ is convex in x for each t;*
(b) *$f_i(x, t)$ is submodular in (x, t); and*
(c) $\text{argmin}_{\{z:z\geq x\}}(c_i(z-x, t)+L_i(z, t)+\gamma\int f_{i+1}(z-w, t)dF_{t,i}(w))$ *is increasing in (x, t).*

Proof. Part (a) follows from (3.10.1), the assumptions that $c_i(y, t)$ and $L_i(z, t)$ are convex in y and in z respectively for each (t, i), the preservation of convexity under the minimization operation (Dantzig [1955]), and induction.

Part (b) trivially holds for $i = m + 1$. Pick any integer i' with $1 \leq i' \leq m$, and suppose that part (b) holds for $i = i' + 1$. To complete the proof by induction, it suffices to show that part (b) and part (c) hold for $i = i'$. By part (a), the induction hypothesis, and Corollary 3.10.1, $\int f_{i'+1}(z - w, t)dF_{t,i'}(w)$ is submodular in (z, t) on $R^1 \times T$. Because $c_{i'}(y, t)$ is convex in y and independent of t and $L_{i'}(z, t)$ is submodular in (z, t) and by part (b) of Lemma 2.6.2 and part (b) of Lemma 2.6.1, $c_{i'}(z - x, t) + L_{i'}(z, t) + \gamma \int f_{i'+1}(z - w, t)dF_{t,i'}(w)$ is submodular in (z, x, t). Part (b) for $i = i'$ follows from (3.10.1) and Theorem 2.7.6, and part (c) for $i = i'$ follows from Theorem 2.8.2. □

For the rest of this subsection, suppose that the production cost is proportional to the level of production so $c_i(y, t) = c_{t,i} \cdot y$ for some $c_{t,i}$ in R^m. Let $g_i(x, t) = f_i(x, t) + c_{t,i} \cdot x$. Define $g_{m+1}(x, t) = 0$ for each (x, t) and $c_{t,m+1} = 0$ for each t. Then (3.10.1) becomes

$$\begin{aligned} g_i(x, t) = \min_{\{z:z\geq x\}}((c_{t,i} - \gamma c_{t,i+1}) \cdot z + L_i(z, t) \\ + \gamma \int g_{i+1}(z - w, t)dF_{t,i}(w) \\ + \gamma c_{t,i+1} \cdot \int w dF_{t,i}(w)). \end{aligned} \tag{3.10.2}$$

Theorem 3.10.3. *Suppose that there is a single product (so $m = 1$), the unit production cost satisfies $c_{t,i} - \gamma c_{t,i+1}$ decreasing in t for each i, the combined expected holding and penalty cost function $L_i(z, t)$ is convex in z and submodular in (z, t) for each i, and the distribution function $F_{t,i}(w)$ for demand is stochastically increasing in t for each i. For each i,*
(a) *$g_{t,i}(x)$ is convex in x for each t;*
(b) *$g_{t,i}(x)$ is submodular in (x, t); and*

(c) $\text{argmin}_{\{z:z\geq x\}}(c_{t,i}z + L_i(z,t) + \gamma\int f_{i+1}(z-w,t)dF_{t,i}(w))$ *is increasing in* (x,t).

Proof. The proof of part (a) follows by induction as in the proof of part (a) of Theorem 3.10.2.

The remainder of the proof proceeds inductively as in the proof of part (b) and part (c) of Theorem 3.10.2, by supposing that part (b) holds for some $i = i' + 1$ and showing that this implies that $(c_{t,i'} - \gamma c_{t,i'+1})z + L_{i'}(z,t) + \gamma\int g_{i'+1}(z-w,t)dF_{t,i'}(w)$ is submodular in (z,t) with the result then following from Theorem 2.7.6 and Theorem 2.8.2. The only difference in the proof of the present result is that the property of $c_{t,i} - \gamma c_{t,i+1}$ decreasing in t is used to show that $(c_{t,i'} - \gamma c_{t,i'+1})z$ is submodular in (z,t). □

Under the conditions of Theorem 3.10.3, it is well-known that there exist critical levels $x'_{t,i}$ such that if the firm begins period i of problem t with x units of inventory on hand then it is optimal to produce $x \vee x'_{t,i} - x$ units in period i; that is, to produce enough so that the inventory level after production is $x \vee x'_{t,i}$. The critical level $x'_{t,i}$ can be chosen as any element of $\text{argmin}_{z\in R^1}(c_{t,i}z + L_i(z,t) + \gamma\int f_{i+1}(z-w,t)dF_{t,i}(w))$ for each (t,i). Under the hypotheses of Theorem 3.10.3, these critical levels can be selected so that $x'_{t,i}$ is increasing in the parameter t for each period i.

Lemma 3.10.1 gives a specific form of the combined expected holding and penalty cost function $L_i(z,t)$ for which the hypotheses of Theorem 3.10.2 and Theorem 3.10.3 hold. This $L_i(z,t)$ corresponds to an inventory problem with a single product (so $m = 1$), a penalty cost $p_{t,i}$ for each unit of unsatisfied demand at the end of period i of problem t, and a holding cost $h_{t,i}$ for each unit of stock held at the end of period i of problem t.

Lemma 3.10.1. *If z is in R^1, $p_{t,i}$ is nonnegative and increasing in t, $h_{t,i}$ is nonnegative and decreasing in t, $F_{t,i}(w)$ is stochastically increasing in t, and*

$$L_i(z,t) = p_{t,i}\int((w-z)\vee 0)dF_{t,i}(w) + h_{t,i}\int((z-w)\vee 0)dF_{t,i}(w),$$

then $L_i(z,t)$ is convex in z and submodular in (z,t).

Proof. The function $L_i(z,t)$ is convex in z because the maximum of two convex functions is convex and the collection of convex functions on R^1 is a closed convex cone in the vector space of all real-valued functions on R^1.

The submodularity of $L_i(z,t)$ in (z,t) follows from part (b) of Lemma 2.6.1 and from Corollary 3.10.1 because $p_{t,i}((-y)\vee 0)$ and $h_{t,i}(y\vee 0)$ are submodular in (y,t) by Corollary 2.6.1 and are convex in z. □

Theorem 3.10.4. *Suppose that the n-vector of unit production costs satisfies* $c_{t,i} - \gamma c_{t,i+1}$ *decreasing in t for each i, the expected combined holding and penalty cost function* $L_i(z, t)$ *is submodular in* (z, t) *for each i, and the distribution functions* $F_{t,i}(w)$ *of demand do not depend on t. For each i,*
(a) $g_i(x, t)$ *is submodular in* (x, t)*; and*
(b) $\text{argmin}_{\{z:z\geq x\}}(c_{t,i} \cdot z + L_i(z, t) + \gamma \int f_{i+1}(z - w, t)dF_{t,i}(w))$ *is increasing in* (x, t)*.*

Proof. Part (a) trivially holds for $i = m + 1$ because $g_{m+1}(x, t) = 0$ for each (x, t). Pick any integer i' with $1 \leq i' \leq m$ and suppose that part (a) holds for $i = i' + 1$. To complete the proof by induction, it suffices to show that part (a) and part (b) hold for $i = i'$. Because $c_{t,i'} - \gamma c_{t,i'+1}$ is decreasing in t, $(c_{t,i'} - \gamma c_{t,i'+1}) \cdot z$ is submodular in (z, t) on $R^n \times T$ by Corollary 2.6.1. Because $F_{t,i'}(w)$ does not depend on t and $g_{i'+1}(x, t)$ is submodular in (x, t) by the induction hypothesis, $\int g_{i'+1}(z-w, t)dF_{t,i'}(w)$ is submodular in (z, t) by Corollary 2.6.2. Furthermore, $L_{i'}(z, t)$ is submodular in (z, t) by assumption. Therefore, $(c_{t,i'} - \gamma c_{t,i'+1}) \cdot z + L_{i'}(z, t) + \gamma \int g_{i'+1}(z - w, t)dF_{t,i'}(w) + \gamma c_{t,i'+1} \cdot \int w dF_{t,i'}(w)$ is submodular in (z, t) by part (b) of Lemma 2.6.1. Now part (a) follows from Theorem 2.7.6 and part (b) follows from Theorem 2.8.2. □

4

Noncooperative Games

4.1 Introduction

This chapter considers the role of supermodularity and complementarity in noncooperative games. The results primarily involve supermodular games, where the payoff function of each player has properties of supermodularity and increasing differences. This introductory section briefly summarizes subsequent sections of this chapter and gives basic definitions and notation used in later sections of this chapter.

Section 4.2 studies general properties of supermodular games. Under modest regularity conditions, an equilibrium point exists. Certain parameterized collections of supermodular games have the property that equilibrium points for each game increase with the parameter. A consequence of the latter result is that for certain supermodular games each player's strategy in an equilibrium point increases as more players are included in the game. The results of this section rely on an equivalence between equilibrium points and fixed points, on fixed point results of Section 2.5, and on properties of best responses in supermodular games.

Section 4.3 gives two algorithms for seeking an equilibrium point in a supermodular game. The operation of the algorithms corresponds to versions of fictitious play in certain dynamic games. Under conditions slightly stronger than required for the existence result of Section 4.2, each algorithm generates a sequence of feasible joint strategies that converges monotonically to an equilibrium point. Each algorithm has an inherent check that determines whether or not an equilibrium point has been attained. When there are finitely many feasible joint strategies, there are bounds on the number of iterations required by each algorithm to find an equilibrium point and these bounds show the algorithms to be very efficient.

Section 4.4 gives examples of supermodular games. These involve a pricing game with substitute products, a production game with complementary products, a multimarket oligopoly, an arms race game, a trading partner search game, an optimal consumption game involving multiple products and multiple consumers, a facility location game, and a noncooperative game version of the minimum cut problem.

A **noncooperative game** is a triple $(\mathbf{N}, \mathbf{S}, \{\boldsymbol{f_i} : \boldsymbol{i} \in \mathbf{N}\})$ consisting of a nonempty set of **players** N, a set S of feasible joint strategies, and a collection of payoff functions $\{f_i : i \in N\}$ such that the payoff function $f_i(x)$ is defined on S for each player i in N. Unless explicitly stated otherwise, the set of players N is assumed to be finite and to take the form $N = \{1, \ldots, n\}$ where $n = |N|$. (The case where N includes infinitely many players is sometimes considered.) The **strategy** of player i is an m_i-vector x_i. Let $m = \sum_{i=1}^{n} m_i$. A **joint strategy** is an m-vector $x = (x_1, \ldots, x_n)$, composed of the strategies x_i of each of the n players. For a more general set of players N, a **joint strategy** takes the form $\{x_i : i \in N\}$ where x_i is the strategy of player i. (More precisely, strategy and joint strategy as defined herein are a **pure strategy** and a **pure joint strategy**. A **mixed strategy** involves a probability distribution on the set of all strategies for a given player, where the player determines the probability distribution and the ultimate strategy for that player is generated randomly according to that probability distribution. A **mixed joint strategy** involves a probability distribution on the set of all joint strategies, where the ultimate joint strategy is generated randomly according to that probability distribution. Only pure strategies and pure joint strategies are considered in this chapter, and the terms "strategy" and "joint strategy" are taken to mean pure strategy and pure joint strategy.) The set of feasible joint strategies is a given subset S of R^m (or of $\times_{i \in N} R^{m_i}$ for a more general set of players N). (The specification of S in any noncooperative game $(N, S, \{f_i : i \in N\})$ is implicitly taken to connote the division of each feasible joint strategy x into its components x_i for each player i in N.) The **payoff function** for each player i in N is a real-valued function $f_i(x)$ defined on S such that for any feasible joint strategy x player i receives the utility $f_i(x)$.

For any joint strategy x and any player i, let $\boldsymbol{x_{-i}}$ denote the vector of strategies of all players in N except player i. For any joint strategy x, any player i, and any m_i-vector y_i, let $(\boldsymbol{y_i}, \boldsymbol{x_{-i}})$ denote the joint strategy vector with the strategy x_i of player i replaced by y_i in x and the other components of x left unchanged. Then $x = (x_i, x_{-i})$ for any joint strategy x and any player i. The set of feasible strategies for player i given strategies x_{-i} for the other players is denoted

$$\mathbf{S_i(x_{-i})} = \{y_i : (y_i, x_{-i}) \in S\};$$

that is, $S_i(x_{-i})$ is the section of S at x_{-i}. The feasible strategies for any player may depend upon the strategies of the other players. For any player i, let

$$\mathbf{S_{-i}} = \{x_{-i} : S_i(x_{-i}) \text{ is nonempty}\}$$

be the collection of all vectors x_{-i} of strategies for the players other that i such that there is some strategy y_i for player i with (y_i, x_{-i}) being a feasible joint strategy; that is, S_{-i} is the projection of S on the coordinates of the strategies of all players except player i. For any player i, let

$$\mathbf{S_i} = \cup_{x_{-i} \in S_{-i}} S_i(x_{-i})$$

be the set of all strategies for player i that are a component of any feasible joint strategy; that is, S_i is the projection of S on the coordinates of the strategy of player i. Note that x_{-i} is in R^{m-m_i}, (y_i, x_{-i}) is in R^m, $S_i(x_{-i})$ is a subset of R^{m_i}, S_{-i} is a subset of R^{m-m_i}, and S_i is a subset of R^{m_i}. For x in S, define

$$\mathbf{S(x)} = (\times_{i \in N} S_i(x_{-i})) \cap S.$$

Note that $S = \times_{i \in N} S_i$ if and only if $S(x) = S$ for each x in S.

For each vector x_{-i} in S_{-i}, the **best response correspondence** for player i is the set

$$\mathbf{Y_i(x_{-i})} = \text{argmax}_{y_i \in S_i(x_{-i})} f_i(y_i, x_{-i})$$

of all strategies that are optimal for player i given x_{-i}. For x_{-i} in S_{-i}, an element of $Y_i(x_{-i})$ is a **best response** given x_{-i}. A **best response function** for player i is any function on S_{-i} that maps each x_{-i} into $Y_i(x_{-i})$. For each feasible joint strategy x in S and each y in $S(x)$, define

$$\mathbf{g(y, x)} = \textstyle\sum_{i \in N} f_i(y_i, x_{-i}).$$

For each feasible joint strategy x in S, the **best joint response correspondence** is the set

$$\mathbf{Y(x)} = \text{argmax}_{y \in S(x)} g(y, x)$$

of all feasible joint strategies such that the strategy for each player i is feasible given x_{-i} and the sum of the payoffs to the n players is maximized given that player i receives the payoff resulting from using the strategy for player i instead of x_i in x. An element of $Y(x)$ for x in S is a **best joint response** given x. A **best joint response function** is any function on S that maps each x into $Y(x)$.

If $S = \times_{i \in N} S_i$ for arbitrary N (finite or not), then the **best joint response correspondence** is the direct product of the individual players' best response correspondences and a **best joint response function** is any function on S that maps each feasible joint strategy into the direct product of the best response correspondences for the individual players; that is,

$$\mathbf{Y(x)} = \times_{i \in N} Y_i(x_{-i}) = \times_{i \in N} \text{argmax}_{y_i \in S_i} f_i(y_i, x_{-i})$$

for each x in S. These definitions are equivalent to those given in the preceding paragraph when N is finite and $S = \times_{i \in N} S_i$. (The noncooperative game model in this chapter permits the set of feasible strategies for some players to depend on the strategies chosen by the other players, so the set S of feasible joint strategies is not limited to being the product set $\times_{i \in N} S_i$. This generality in the form of S may come at a loss of generality in some other respects, particularly because of the convenience of having the best joint response correspondence equaling the direct product of the individual players' best response correspondences when $S = \times_{i \in N} S_i$ (for arbitrary N). For example, a model with infinitely many players would be difficult to handle since the function $g(y, x)$ used in defining the best joint response correspondence $Y(x)$ may not then be real-valued or even well-defined because it would be constructed as an infinite sum. The case where the set of feasible joint strategies is the direct product of the individual players' sets of feasible strategies is more tractable with infinitely many players because the best joint response correspondence is the direct product of the individual players' best response correspondences. Furthermore, the summation in the definition of $g(y, x)$ fails to preserve properties like quasisupermodularity and the single crossing property from the individual payoff functions, although it does preserve supermodularity and increasing differences. See part (b) of Lemma 2.6.1 and Theorem 2.6.5.) For the case $S = \times_{i \in N} S_i$, the next two sections briefly note extensions of results (holding for more general S) from N being finite to N infinite and from supermodular games to quasisupermodular games (defined below).

The notation x_{-i}, $S_i(x_{-i})$, S_{-i}, S_i, $S(x)$, $Y_i(x_{-i})$, $g(y, x)$, and $Y(x)$ are used in later sections of this chapter as defined above.

A feasible joint strategy x is an **equilibrium point** if

$$f_i(y_i, x_{-i}) \leq f_i(x) \text{ for each } y_i \text{ in } S_i(x_{-i}) \text{ and each } i \text{ in } N;$$

that is, if x is in S and x_i is in the best response correspondence $Y_i(x_{-i})$ for each i. Given an equilibrium point, there is no feasible way for any player to strictly improve its utility if the strategies of all the other players remain unchanged.

A noncooperative game $(N, S, \{f_i : i \in N\})$ is a **supermodular game** if the set S of feasible joint strategies is a sublattice of R^m (or of $\times_{i \in N} R^{m_i}$), the payoff function $f_i(y_i, x_{-i})$ is supermodular in y_i on S_i for each x_{-i} in S_{-i} and each player i, and $f_i(y_i, x_{-i})$ has increasing differences in (y_i, x_{-i}) on $S_i \times S_{-i}$ for each i. These hypotheses on the payoff function for each player i imply that, by Theorem 2.6.1, from each player i's point of view (that is, with respect to that player's payoff function) each pair of components of player i's strategy are complements and each component of player i's strategy is complementary with each component of the strategy of each other player. The assumptions on the payoff function $f_i(y_i, x_{-i})$ for each player i in a supermodular game correspond to the assumptions on the objective function in Theorem 2.8.1. A noncooperative game is a **quasisupermodular game** if the set S of feasible joint strategies is a sublattice of R^m (or of $\times_{i \in N} R^{m_i}$), the payoff function $f_i(y_i, x_{-i})$ is quasisupermodular in y_i on S_i for each x_{-i} in S_{-i} and each player i, and $f_i(y_i, x_{-i})$ satisfies the single crossing property in (y_i, x_{-i}) on $S_i \times S_{-i}$ for each i. The assumptions on the payoff function $f_i(y_i, x_{-i})$ for each player i in a quasisupermodular game correspond to the assumptions on the objective function in Theorem 2.8.6.

4.2 Existence of an Equilibrium Point, Parametric Properties

This section establishes the existence of equilibrium points in supermodular games with modest regularity conditions and gives conditions for equilibrium points to increase with respect to the parameter in a parameterized collection of supermodular games. The penultimate paragraph of this section notes extensions of these results to quasisupermodular games and games that may have infinitely many players.

The following well-known result characterizes the equilibrium points of a noncooperative game as the fixed points of the best joint response correspondence.

Lemma 4.2.1. *The set of all equilibrium points for a noncooperative game* $(N, S, \{f_i : i \in N\})$ *is identical to the set of fixed points for the best joint response correspondence* $Y(x)$ *for* x *in* S.

Proof. Suppose that x' is a fixed point of $Y(x)$ on S. Pick any player i and any y_i in $S_i(x'_{-i})$. Then (y_i, x'_{-i}) is in $S(x')$, and so

$$0 \le g(x', x') - g((y_i, x'_{-i}), x') = f_i(x') - f_i(y_i, x'_{-i})$$

where the inequality holds because x' is in $Y(x')$. Therefore, x' is an equilibrium point.

Now suppose that x' is an equilibrium point. Pick any y in $S(x')$, so y_i is in $S_i(x'_{-i})$ for each i. Because x' is an equilibrium point, $f_i(y_i, x'_{-i}) \le f_i(x')$ for each i and so

$$\begin{aligned} 0 &\le \textstyle\sum_{i \in N}(f_i(x') - f_i(y_i, x'_{-i})) = \sum_{i \in N} f_i(x') - \sum_{i \in N} f_i(y_i, x'_{-i}) \\ &= g(x', x') - g(y, x'). \end{aligned}$$

Therefore, x' is a fixed point of $Y(x) = \text{argmax}_{y \in S(x)}\, g(y, x)$. $\square$

For a supermodular game with certain regularity conditions, Lemma 4.2.2 shows that each set of best responses and each set of best joint responses is a nonempty compact sublattice having a greatest element and a least element, that the best response correspondences and the best joint response correspondence are increasing with their arguments, and that the greatest and least elements from each correspondence for each argument are also increasing with the argument. These results for the best joint response correspondence are from Topkis [1979].

Lemma 4.2.2. *Consider a supermodular game $(N, S, \{f_i : i \in N\})$ for which the set S of feasible joint strategies is nonempty and compact and the payoff function $f_i(y_i, x_{-i})$ is upper semicontinuous in y_i on $S_i(x_{-i})$ for each x_{-i} in S_{-i} and each i.*
(a) *The set $Y_i(x_{-i})$ of best responses for each player i is a nonempty compact sublattice of R^{m_i} for each x_{-i} in S_{-i}.*
(b) *The set $Y(x)$ of best joint responses is a nonempty compact sublattice of R^m for each x in S.*
(c) *There exists a greatest and a least best response for each player i and each x_{-i} in S_{-i}; that is, each $Y_i(x_{-i})$ has a greatest element and a least element.*
(d) *There exists a greatest and a least best joint response for each x in S; that is, each $Y(x)$ has a greatest element and a least element.*
(e) *The best response correspondence $Y_i(x_{-i})$ is increasing in x_{-i} on S_{-i} for each player i.*
(f) *The best joint response correspondence $Y(x)$ is increasing in x on S.*
(g) *The greatest (least) best response (that is, the greatest (least) element of $Y_i(x_{-i})$) is an increasing function from S_{-i} into S_i for each player i.*
(h) *The greatest (least) best joint response (that is, the greatest (least) element of $Y(x)$) is an increasing function from S into S.*

Proof. Since S is compact, its section $S_i(x_{-i})$ is a compact subset of R^{m_i} for each i and each x_{-i} in S_{-i}. If x_{-i} is in S_{-i}, then $S_i(x_{-i})$ is nonempty by the definition of S_{-i}. Therefore, $Y_i(x_{-i}) = \text{argmax}_{y_i \in S_i(x_{-i})} f_i(y_i, x_{-i})$ is nonempty and compact because it is the set of maxima of an upper semicontinuous function on a nonempty compact set. Because S is a sublattice of R^m, its section $S_i(x_{-i})$ is a sublattice of R^{m_i} for each i and each x_{-i} in S_{-i} by part (a) of Lemma 2.2.3. Part (a) and part (c) now follow from Corollary 2.7.1, because $f_i(y_i, x_{-i})$ is supermodular in y_i on $S_i(x_{-i})$.

Since S is compact, $S(x)$ is a compact subset of R^m for each x in S. If x is in S then x is in $S(x)$, so $S(x)$ is nonempty for each x in S. Because $f_i(y_i, x_{-i})$ is upper semicontinuous in y_i on $S_i(x_{-i})$ for each i and x_{-i}, $g(y, x)$ is upper semicontinuous in y on $S(x)$. Therefore, $Y(x)$ is nonempty and compact because it is the set of maxima of an upper semicontinuous function on a nonempty compact set. Because S is a sublattice of R^m, $S(x)$ is sublattice of R^m for each x in S by part (a) of Lemma 2.2.3, part (d) of Example 2.2.5, and Lemma 2.2.2. Because $f_i(y_i, x_{-i})$ is supermodular in y_i on S_i for each x_{-i} in S_{-i}, $g(y, x)$ is supermodular in y on $\times_{i \in N} S_i$ by part (b) of Lemma 2.6.1 and hence is supermodular in y on $S(x)$ for each x in S. Part (b) and part (d) now follow from Corollary 2.7.1.

Because S is a sublattice of R^m, the section $S_i(x_{-i})$ is increasing in x_{-i} on the projection S_{-i} for each i by part (a) of Theorem 2.4.5. Because $f_i(y_i, x_{-i})$ is supermodular in y_i on S_i and has increasing differences in (y_i, x_{-i}) on $S_i \times S_{-i}$, part (e) and part (g) follow from Theorem 2.8.1 and part (a) of Theorem 2.8.3.

Because S is a sublattice of R^m, $S(x)$ is increasing in x on S by part (a) of Theorem 2.4.5, Example 2.4.2, and Theorem 2.4.2. Because $f_i(y_i, x_{-i})$ has increasing differences in (y_i, x_{-i}) on $S_i \times S_{-i}$, $g(y, x)$ has increasing differences in (y, x) on $S \times S$. Recalling that $g(y, x)$ is supermodular in y on $\times_{i \in N} S_i$ and hence on S for each x in S, part (f) and part (h) now follow from Theorem 2.8.1 and part (a) of Theorem 2.8.3. □

Theorem 4.2.1 shows that the set of equilibrium points for a supermodular game with some regularity conditions is a nonempty complete lattice and there exist a greatest and a least equilibrium point. Topkis [1979] gives the part of this result establishing the existence of some equilibrium point and, indeed, a greatest and a least equilibrium point. Zhou [1994] further shows that the set of equilibrium points is a complete lattice. The proof of Theorem 4.2.1 is based on the property that the set $Y(x)$ of best joint responses is increasing in the joint strategy x by part (f) of Lemma 4.2.2, so the properties of a supermodular game imply that the conditions of Theorem 2.5.1 hold and one can apply Lemma 4.2.1. Algorithm 4.3.1 and Algorithm 4.3.2 in Section 4.3 use, respectively, the best response functions and the best joint response function to constructively approximate the least (and the greatest) equilibrium point of certain supermodular games, thereby proving the existence of a least (and a greatest) equilibrium point. The proof of Theorem 4.2.1 is not constructive, but the regularity conditions among its hypotheses are more general that those required to establish the validity of Algorithm 4.3.1 and Algorithm 4.3.2. Furthermore, Theorem 4.2.1 yields the additional result that the set of equilibrium points is a complete lattice.

Theorem 4.2.1. *If $(N, S, \{f_i : i \in N\})$ is a supermodular game, the set S of feasible joint strategies is nonempty and compact, and the payoff function $f_i(y_i, x_{-i})$ is upper semicontinuous in y_i on $S_i(x_{-i})$ for each x_{-i} in S_{-i} and each i, then the set of equilibrium points is a nonempty complete lattice and a greatest and a least equilibrium point exist.*

Proof. By Theorem 2.3.1, S is a subcomplete sublattice of R^m and so S is a complete lattice. By part (f) of Lemma 4.2.2, the best joint response correspondence $Y(x)$ is increasing from S into $\mathscr{L}(S)$. By part (b) of Lemma 4.2.2, $Y(x)$ is a compact sublattice of R^m for each x in S and so $Y(x)$ is a subcomplete sublattice of R^m by Theorem 2.3.1 and consequently $Y(x)$ is a subcomplete sublattice of S for each x in S. By part (b) of Theorem 2.5.1, the set of fixed points of the best joint response correspondence $Y(x)$ is a nonempty complete lattice and has a greatest element and a least element. The result now follows from Lemma 4.2.1 because the set of equilibrium points is identical to the set of fixed points of $Y(x)$ in S. □

When $n = 1$, the set of equilibrium points is simply the set of all maxima of $f_1(x)$ subject to x in S. If $n = 1$ and the hypotheses of Theorem 4.2.1 hold, part (b) of Corollary 2.7.1 implies that the set of equilibrium points is a compact sublattice of R^m. Example 4.2.1 and Example 4.2.2 show, however, that when $n > 1$ and the hypotheses of Theorem 4.2.1 hold, the set of equilibrium points need not be compact and need not be a sublattice. (Even without the supermodular game's special assumptions about the set of feasible joint strategies and the payoff functions, the set of equilibrium points is compact under the other hypotheses stated in Theorem 4.3.2 by a proof similar to the first paragraph of the proof of Theorem 4.3.2. In Example 4.2.1, $S_i(x_{-i})$ is not a lower semicontinuous correspondence as required in the assumptions of Theorem 4.3.2.)

Example 4.2.1. Consider a noncooperative game $(N, S, \{f_i : i \in N\})$ with $n = 2$, $m_1 = m_2 = 1$, $S = \{x : x_1 = x_2, x_2 \in [0, 1)\} \cup ([1, 2] \times [1, 2])$, $f_1(x) = x_1$, and $f_2(x) = x_2$. This is a supermodular game and the other hypotheses of Theorem 4.2.1 are satisfied. The set of equilibrium points is $\{(z, z) : z \in [0, 1) \cup \{2\}\}$, which is not closed. (But as in Theorem 4.2.1, the set of equilibrium points is a complete lattice. Indeed, it is a complete sublattice of R^2, but it is not subcomplete.)

Example 4.2.2. Consider a noncooperative game $(N, S, \{f_i : i \in N\})$ with $n = 3$, $m_1 = m_2 = m_3 = 1$, $S = \times_{i=1}^3 [0, 1]$, and $f_1(x) = f_2(x) = f_3(x) = x_1 x_2 x_3$. This is a supermodular game, and the other hypotheses of Theorem 4.2.1 are satisfied. The set of equilibrium points is $\{(z, 0, 0) : z \in [0, 1]\} \cup \{(0, z, 0) : z \in [0, 1]\} \cup \{(0, 0, z) : z \in [0, 1]\} \cup \{(1, 1, 1)\}$, which is a lattice. But the set of equilibrium points is not a sublattice of R^3, since $(1, 0, 0)$ and $(0, 1, 0)$ are equilibrium points but $(1, 1, 0) = (1, 0, 0) \vee (0, 1, 0)$ is not an equilibrium point.

A feasible joint strategy x for a noncooperative game $(N, S, \{f_i : i \in N\})$ is a **strong equilibrium point** if there does not exist a subset N' of the players

N such that the sum of the payoff functions to the players in N' can be strictly increased with a different feasible joint strategy that leaves unchanged the strategies of the players not in N'. Example 4.2.3 shows that a strong equilibrium point may not exist even for a supermodular game with $n = 2$, $m_1 = m_2 = 1$, and each payoff function being concave.

Example 4.2.3. Consider a noncooperative game $(N, S, \{f_i : i \in N\})$ with $n = 2$, $m_1 = m_2 = 1$, $S = [0, 1] \times [0, 1]$, $f_1(x) = -x_1^2 + x_2$, and $f_2(x) = -x_2^2 + x_1$. Then $(0, 0)$ is the unique equilibrium point, but players 1 and 2 can strictly increase the sum of their individual payoffs with the joint strategy $(1/2, 1/2)$.

Theorem 4.2.2 gives conditions on a parameterized collection of supermodular games such that the greatest (least) equilibrium point for the game corresponding to each particular parameter increases with the parameter. Milgrom and Roberts [1990a] and Sobel [1988] independently establish versions of this result, which is related to a more abstract approach of Lippman, Mamer, and McCardle [1987].

Theorem 4.2.2. *Suppose that T is a partially ordered set and $(N, S^t, \{f_i^t : i \in N\})$ is a collection of supermodular games parameterized by t in T where in game t the payoff function for each player i is $f_i^t(x)$ and the set of feasible joint strategies is S^t. The set S^t of feasible joint strategies is nonempty and compact for each t in T and is increasing in t on T. Let S_{-i}^t and $S_i^t(x_{-i})$ denote the dependence of S_{-i} and $S_i(x_{-i})$ on the parameter t. For each player i and each x_{-i} in S_{-i}^t, the payoff function $f_i^t(y_i, x_{-i})$ is upper semicontinuous in y_i on $S_i^t(x_{-i})$ for each t in T and has increasing differences in (y_i, t) on $(\cup_{t \in T} S_i^t) \times T$. Then there exists a greatest equilibrium point and a least equilibrium point for each game t in T, and the greatest (least) equilibrium point for game t is increasing in t on T.*

Proof. For t in T and x in S^t, let $Y(x, t)$ be the best joint response correspondence for game t. By part (b) of Lemma 4.2.2, $Y(x, t)$ is a nonempty compact sublattice of R^m for all t in T and x in S^t. By Theorem 2.3.1, each $Y(x, t)$ is a subcomplete sublattice of R^m. By Theorem 2.8.1 (using also part (a) of Theorem 2.4.5, Example 2.4.2, and Theorem 2.4.2), $Y(x, t)$ is increasing in (x, t) on $\cup_{t \in T}(S^t \times \{t\})$. (Part (f) of Lemma 4.2.2 already shows that $Y(x, t)$ is increasing in x on S^t for each t in T.) Part (a) and part (b) of Theorem 2.5.2 imply that $Y(x, t)$ has a greatest (least) fixed point for each t in T and that the greatest (least) fixed point is increasing in t on T. The result now follows because the set of fixed points for $Y(x, t)$ given t in T is identical to the set of equilibrium points for game t by Lemma 4.2.1. □

Theorem 4.2.3 gives conditions such that the strategy of each player in the greatest (least) equilibrium point of a supermodular game increases as additional players are included in the game.

Theorem 4.2.3. *Suppose that* $(N, S, \{f_i : i \in N\})$ *is a supermodular game, the set S of feasible joint strategies is nonempty and compact, and the payoff function* $f_i(y_i, x_{-i})$ *is upper semicontinuous in* y_i *on* $S_i(x_{-i})$ *for each player i and each* x_{-i} *in* S_{-i}. *For each x in S and each subset N′ of N, let* $x_{N'} = \{x_i : i \in N'\}$. *Let x′ be the least element of S. For each subset N′ of N, let* $S^{N'}$ *be the section of S at* $x'_{N\setminus N'}$ (*that is, the set of all* $x_{N'}$ *such that* $(x_{N'}, x'_{N\setminus N'})$ *is in S*). *For each subset N′ of N, each player i in N′, and each* $x_{N'}$ *in* $S^{N'}$, *let* $f_i^{N'}(x_{N'}) = f_i(x_{N'}, x'_{N\setminus N'})$. *Consider the collection of supermodular games* $(N', S^{N'}, \{f_i^{N'} : i \in N'\})$ *parameterized by the nonempty subsets N′ of N. Then there exists a greatest equilibrium point and a least equilibrium point for each game N′, and for each player i the strategy of player i in the greatest* (*least*) *equilibrium point for game N′ is increasing in N′ where i is included in N′.*

Proof. The least element x' of S exists by Corollary 2.3.2. For each nonempty subset N' of N, the equilibrium points of the game $(N', S^{N'}, \{f_i^{N'} : i \in N'\})$ are the vectors of the strategies of the players N' from the equilibrium points of the game $(N, \{x : x \in S, x_{N\setminus N'} = x'_{N\setminus N'}\}, \{f_i : i \in N\})$. Because $\{x : x \in S, x_{N\setminus N'} = x'_{N\setminus N'}\}$ is increasing in N' by part (b) of Example 2.4.1, the result follows from Theorem 4.2.2. □

Now consider a noncooperative game $(N, S, \{f_i : i \in N\})$ where the set S of feasible joint strategies takes the form $S = \times_{i\in N} S_i$, so $Y(x) = \times_{i\in N} Y_i(x_{-i}) = \times_{i\in N} \text{argmax}_{y_i \in S_i} f_i(y_i, x_{-i})$. If the set of players N is arbitrary (that is, not necessarily finite) and if the games are quasisupermodular games rather than supermodular games, then versions of the parts of Lemma 4.2.2 for best joint responses and for the best joint response correspondence hold (where the extension to quasisupermodular games involves using Theorem 2.7.2 and Theorem 2.8.6 instead of Corollary 2.7.1, Theorem 2.8.1, and Theorem 2.8.3). (Versions of the parts of Lemma 4.2.2 for best responses and for the best response correspondences hold for infinitely many players and for quasisupermodular games even without the additional assumption that $S = \times_{i\in N} S_i$.) Versions of Theorem 4.2.1 and Theorem 4.2.2 follow similarly as above from the extended versions of Lemma 4.2.2 for games with infinitely many players (as shown by Milgrom and Roberts [1990a]) and for quasisupermodular games (as shown by Milgrom and Shannon [1994]). A version of Theorem 4.2.3 for games with infinitely many players and for quasisupermodular games likewise follows similarly.

Other theory related to supermodular games is given in Fudenberg and Tirole [1991], Kandori and Rob [1995], Milgrom and Roberts [1990a, 1991], Milgrom and Shannon [1994], and Vives [1990].

4.3 Algorithms for Approximating an Equilibrium Point

Algorithm 4.3.1 and Algorithm 4.3.2 approximate an equilibrium point for a supermodular game based on versions of fictitious play in sequential games corresponding to natural behavioral processes. (Extensions to a quasisupermodular game and, for the second algorithm, to infinitely many players are also noted.) Algorithm 4.3.1, considered in Subsection 4.3.1, corresponds to the iterative decision-making process by which the n players take turns with each player successively maximizing that player's own payoff function with respect to its own feasible strategies while the strategies of the other $n-1$ players are held fixed; that is, each individual player proceeds in round-robin fashion to update its own strategy by selecting a best response. Algorithm 4.3.2, considered in Subsection 4.3.2, corresponds to the iterative decision-making process by which the n players concurrently pick a feasible joint strategy such that the new strategy for each player is feasible given the previous strategies for the other players and the sum of the payoff functions to the n players is maximized where each player receives the payoff function resulting from replacing its previous strategy with its new strategy in the previous joint strategy; that is, the new joint strategy is a best joint response to the previous joint strategy. (As noted following the formal statement of Algorithm 4.3.2 in Subsection 4.3.2, if each player's set of feasible strategies is independent of the strategies of the other players then this process reduces to having each player individually and simultaneously select a best response to the previous joint strategy with the new joint strategy composed of the new strategies for each of the players.)

Based on a suggestion of Brown [1951], Robinson [1951] proves that related versions of fictitious play can be used to generate sequences of feasible (pure) joint strategies for finite zero-sum two-player games such that the frequencies of past strategies for each player yield feasible mixed joint strategies converging to a (mixed joint strategy) equilibrium point. However, even for very simple games the (pure) joint strategies generated by these processes need not generally converge. The present section, based on results in Topkis [1979], shows that versions of fictitious play have strong convergence properties for approximating an equilibrium point in supermodular games.

4.3.1 Round-Robin Optimization

Algorithm 4.3.1 formally describes the procedure for round-robin optimization by the players.

Algorithm 4.3.1. Given a noncooperative game $(N, S, \{f_i : i \in N\})$, proceed as follows to generate a (finite or infinite) sequence of feasible joint strategies.
(a) If S has a least element $\inf(S)$, set $x^{0,0} = \inf(S)$. Otherwise, stop.
(b) Given $x^{k,i}$ in S for any nonnegative integers k and i with $i < n$, let $x^{k,i+1} = (x_{i+1}^{k,i+1}, x_{-(i+1)}^{k,i})$ where $x_{i+1}^{k,i+1}$ is the least element of the best response correspondence $Y_{i+1}(x_{-(i+1)}^{k,i})$ if such an element exists. Otherwise, stop.
(c) Set $i = i + 1$. If $i = n$ so $x^{k,n}$ has been generated for some k, set $x^{k+1,0} = x^{k,n}$, set $k = k + 1$, and set $i = 0$. Return to step (b) and continue.

Theorem 4.3.1 establishes monotonicity, stopping, and efficiency properties for Algorithm 4.3.1 when applied to a supermodular game with certain regularity conditions. Part (a) of Theorem 4.3.1 shows that the sequence $x^{k,i}$ generated by Algorithm 4.3.1 is increasing in k and i. This reduces the problem of finding $x^{k,i+1}$ given $x^{k,i}$ for $i < n$ in step (b) of Algorithm 4.3.1 from a maximization problem over $S_{i+1}(x_{-(i+1)}^{k,i})$ to a maximization problem over $S_{i+1}(x_{-(i+1)}^{k,i}) \cap [x_{i+1}^{k,i}, \infty)$. Part (b) of Theorem 4.3.1 shows that if a joint strategy generated by step (b) of Algorithm 4.3.1 is identical with the joint strategies generated by step (b) in each of the preceding $n - 1$ iterations, then this joint strategy is an equilibrium point. This result can be incorporated into a stopping rule whereby step (c) of Algorithm 4.3.1 would terminate the algorithm after the same joint strategy has been generated n successive times by step (b), as this joint strategy must be an equilibrium point. Furthermore, part (c) of Theorem 4.3.1 shows that if Algorithm 4.3.1 ever generates an equilibrium point, then it continues to generate that same equilibrium point at all subsequent iterations of step (b). Hence, a joint strategy generated in step (b) of Algorithm 4.3.1 is an equilibrium point if and only if it is generated n successive times in part (b). Part (d) of Theorem 4.3.1 indicates that Algorithm 4.3.1 is very efficient computationally when S is finite.

Theorem 4.3.1. *Consider a supermodular game* $(N, S, \{f_i : i \in N\})$ *for which the set* S *of feasible joint strategies is nonempty and compact and the payoff function* $f_i(y_i, x_{-i})$ *is upper semicontinuous in* y_i *on* $S_i(x_{-i})$ *for each* x_{-i} *in* S_{-i} *and each* i.
(a) *Algorithm* 4.3.1 *never stops at step* (a) *or step* (b) *and Algorithm* 4.3.1 *generates an infinite sequence* $x^{k,i}$ *that is increasing in* k *and* i *for* $k = 0, 1, \ldots$ *and* $i = 0, \ldots, n$. *Hence, there exists a feasible joint strategy* x' *in* S *such that* $\lim_{k\to\infty} x^{k,i} = x'$ *for* $i = 0, \ldots, n$.
(b) *If a feasible joint strategy appears* n *successive times in the sequence* $\{x^{k,i} : k \geq 0, 1 \leq i \leq n\}$ *generated by step* (b) *of Algorithm* 4.3.1, *then it is an equilibrium point.*

(c) *If Algorithm* 4.3.1 *generates an equilibrium point at some iteration, then Algorithm* 4.3.1 *generates that same equilibrium point at each subsequent iteration.*
(d) *If, in addition, S has a finite number of elements and q_i is an upper bound on the number of elements in any chain contained in S_i for $i = 1, \ldots, n$, then Algorithm* 4.3.1 *generates an equilibrium point in no more that* $(n-1)(\sum_{i=1}^{n} q_i) - n^2 + n + 1$ *iterations of step* (b).

Proof. Algorithm 4.3.1 does not stop at step (a) because the nonempty compact sublattice S has a least element by Corollary 2.3.2. Algorithm 4.3.1 does not stop at step (b) by part (c) of Lemma 4.2.2. Therefore, step (a) and step (b) of Algorithm 4.3.1 proceed without stopping, and Algorithm 4.3.1 generates an infinite sequence of feasible joint strategies.

Since $x^{0,0}$ is the least element of the sublattice S, $x^{0,i} \le x^{0,i+1}$ for $i = 0, \ldots, n-1$. Now suppose that integers k' and i' are such that $1 \le k'$, $0 \le i' \le n-1$, $x^{k,i} \le x^{k,i+1}$ for all $k = 0, \ldots, k'-1$ and $i = 0, \ldots, n-1$, and $x^{k',i} \le x^{k',i+1}$ for $i = 0, \ldots, i'-1$. Since this supposition holds for $k' = 1$ and $i' = 0$, it suffices to show that $x^{k',i'} \le x^{k',i'+1}$ for the proof of part (a) to follow by induction. Because $x_{i'+1}^{k'-1,i'+1}$ is the least element of $Y_{i'+1}(x_{-(i'+1)}^{k'-1,i'})$, $x_{i'+1}^{k',i'+1}$ is the least element of $Y_{i'+1}(x_{-(i'+1)}^{k',i'})$, and $x^{k'-1,i'} \le x^{k',i'}$ by the induction hypothesis, part (g) of Lemma 4.2.2 implies that $x_{i'+1}^{k'-1,i'+1} \le x_{i'+1}^{k',i'+1}$ and hence

$$x^{k',i'} = (x_{i'+1}^{k'-1,i'+1}, x_{-(i'+1)}^{k',i'}) \le (x_{i'+1}^{k',i'+1}, x_{-(i'+1)}^{k',i'}) = x^{k',i'+1}.$$

The proof of part (a) is completed by noting that an increasing sequence has a limit point in a compact set.

Part (b) follows directly from the definition of an equilibrium point.

Suppose $x^{k,i}$ is an equilibrium point and $i < n$. To establish part (c), it suffices to show that $x^{k,i+1} = x^{k,i}$ or, equivalently, that $x_{i+1}^{k,i+1} = x_{i+1}^{k,i}$. Because $x^{k,i}$ is an equilibrium point, $x_{i+1}^{k,i}$ is in $Y_{i+1}(x_{-(i+1)}^{k,i})$. By step (b) of Algorithm 4.3.1, $x_{i+1}^{k,i+1}$ is the least element of $Y_{i+1}(x_{-(i+1)}^{k,i})$ and so $x_{i+1}^{k,i+1} \le x_{i+1}^{k,i}$. By part (a), $x_{i+1}^{k,i} \le x_{i+1}^{k,i+1}$. Hence, $x_{i+1}^{k,i+1} = x_{i+1}^{k,i}$ and part (c) holds.

Now include the additional hypotheses of part (d). By part (a), $x_h^{k,i}$ is increasing in k and i for $k = 0, 1, \ldots, i = 0, \ldots, n$, and any fixed $h = 1, \ldots, n$. Thus as Algorithm 4.3.1 proceeds, $x_h^{k,i}$ can change its value no more than $q_h - 1$ times and $x^{k,i}$ can change its value no more than $\sum_{h=1}^{n}(q_h - 1)$ times and the sequence $\{x^{k,i}\}$ can contain no more than $\sum_{h=1}^{n}(q_h - 1) + 1$ distinct elements. Since $x^{k,i}$ is increasing in k and i and S is finite, Algorithm 4.3.1 must eventually generate some feasible joint strategy at some iteration such that the same feasible joint strategy is generated at all subsequent iterations.

By part (b), that last distinct feasible joint strategy generated must be an equilibrium point. Since $x^{k,i}$ is increasing in k and i all appearances of a feasible joint strategy in this sequence must be consecutive, and by part (b) and part (c) no feasible joint strategy except the last distinct one generated can appear in $\{x^{k,i} : k \geq 0, 1 \leq i \leq n\}$ more that $n - 1$ times. Thus Algorithm 4.3.1 must attain an equilibrium point in no more than $(n-1)\sum_{h=1}^{n}(q_h - 1) + 1$ iterations of step (b). □

Consider the efficiency implications of the bound in part (d) of Theorem 4.3.1. If S is finite, $S_i(x_{-i}) = S_i$ for each x_{-i} in S_{-i} and each i, and S_i has q_i elements for $i = 1, \ldots, n$, then adding one element to $S_{i'}$ for some player i' adds $\prod_{i \neq i'} q_i$ elements to S but the bound on the number of iterations required by Algorithm 4.3.1 only increases by either $n - 1$ or 0. If S is finite, $S_i(x_{-i}) = S_i$ for each x_{-i} in S_{-i} and each i, and q is an upper bound on the number of elements contained in any chain in S_i for each i, then the bound established in part (d) of Theorem 4.3.1 on the number of iterations required by step (b) of Algorithm 4.3.1 is $(n^2 - n)(q - 1) + 1$, which increases with n^2, while the upper bound q^n on the number of feasible joint strategies increases exponentially with the number of players n. These observations and Example 4.3.1 illustrate the efficiency of Algorithm 4.3.1 for S finite.

Example 4.3.1. Consider a supermodular game $(N, S, \{f_i : i \in N\})$ with $n = 20$, $S_i = \{1, \ldots, 10\}$ for each i, and $S = \times_{i=1}^{n} S_i$. For each i, let $q_i = 10$ so $q_i = |S_i|$ is an upper bound on the number of elements in any chain in S_i. The set S contains $\prod_{i=1}^{n} q_i = 10^{20}$ feasible joint strategies. However, Algorithm 4.3.1 generates an equilibrium point in no more than $(n - 1)(\sum_{i=1}^{n} q_i) - n^2 + n + 1 = (19)(20)(10) - (20)^2 + 20 + 1 = 3421$ iterations of step (b) by part (d) of Theorem 4.3.1. Since each iteration of step (b) requires looking at no more than $q_i = 10$ joint strategies (for some i) and making at most $q_i - 1 = 9$ comparisons (for some i), one can find an equilibrium point for this noncooperative game with 10^{20} feasible joint strategies by looking at no more than $(10)(3421) = 34,210$ joint strategies and making no more than $9(3421) = 30,789$ comparisons.

A correspondence $Z(x)$ from a subset X of R^p into nonempty subsets of R^r is a **lower semicontinuous correspondence** if for any sequence $\{x^k : k = 1, 2, \ldots\}$ in X with limit point x' in X and any z' in $Z(x')$ there exists a sequence $\{z^k : k = 1, 2, \ldots\}$ with z^k in $Z(x^k)$ for each k and having limit point z'. For a noncooperative game $(N, S, \{f_i : i \in N\})$, if the set of feasible joint strategies S has a finite number of elements or if $S = \times_{i=1}^{n} S_i$ where S_i is contained in R^{m_i} for each i (so that $S_i(x_{-i}) = S_i$ for each x_{-i} in S_{-i} and each i), then $S_i(x_{-i})$ is a lower semicontinuous correspondence from S_{-i} into subsets of R^{m_i} for each i and $S(x)$ is a lower semicontinuous correspondence from S into subsets of R^m.

Theorem 4.3.2 gives conditions for the limit point of the sequence generated by Algorithm 4.3.1 to be an equilibrium point. Thus Theorem 4.3.2 constructively establishes the existence of an equilibrium point for supermodular games (with certain regularity conditions), while the existence proof of Theorem 4.2.1 is not constructive. However, the hypotheses of Theorem 4.3.2 are stronger than those of Theorem 4.2.1 (and of Theorem 4.3.1). The additional assumptions of Theorem 4.3.2 are that each $S_i(x_{-i})$ is a lower semicontinuous correspondence and that each payoff function $f_i(x)$ is continuous in x on S (rather than just being upper semicontinuous in x_i given any x_{-i}). Example 4.3.2 and Example 4.3.3 show that these additional assumptions cannot be dispensed with entirely.

Theorem 4.3.2. *Consider a supermodular game* $(N, S, \{f_i : i \in N\})$ *for which the set* S *of feasible joint strategies is nonempty and compact,* $S_i(x_{-i})$ *is a lower semicontinuous correspondence from* S_{-i} *into subsets of* R^{m_i} *for each* i*, and the payoff function* $f_i(x)$ *is continuous in* x *on* S *for each* i*. The limit point of the increasing sequence generated by Algorithm* 4.3.1 *is an equilibrium point. Furthermore, it is the least equilibrium point.*

Proof. Let $\{x^{k,i}\}$ be the sequence generated by Algorithm 4.3.1. By part (a) of Theorem 4.3.1, this sequence is increasing in k and i and it converges to a limit point x'. Pick any i with $1 \leq i \leq n$ and any y_i' in $S_i(x_{-i}')$. Since $\lim_{k\to\infty} x^{k,i} = x'$ and $S_i(x_{-i})$ is a lower semicontinuous correspondence, there exists y_i^k in $S_i(x_{-i}^{k,i})$ for $k = 0, 1, \ldots$ such that $\lim_{k\to\infty} y_i^k = y_i'$. By the construction of $x_i^{k,i}$ in step (b) of Algorithm 4.3.1, $f_i(y_i^k, x_{-i}^{k,i}) \leq f_i(x_i^{k,i}, x_{-i}^{k,i}) = f_i(x^{k,i})$ for each k. By the continuity of $f_i(x)$,

$$f_i(y_i', x_{-i}') = \lim_{k\to\infty} f_i(y_i^k, x_{-i}^{k,i}) \leq \lim_{k\to\infty} f_i(x^{k,i}) = f_i(x').$$

Hence, x' is an equilibrium point.

Let x'' be any equilibrium point. Since $x^{0,0}$ is the least element of S by step (a) of Algorithm 4.3.1, $x^{0,0} \leq x''$. Now suppose that $x^{k,i} \leq x''$ for some $k \geq 0$ and $0 \leq i < n$. Since $x_{i+1}^{k,i+1}$ is the least element of $Y_{i+1}(x_{-(i+1)}^{k,i})$ by the construction in step (b) of Algorithm 4.3.1, x_{i+1}'' is in $Y_{i+1}(x_{-(i+1)}'')$ because x'' is an equilibrium point, and $x^{k,i} \leq x''$, part (g) of Lemma 4.2.2 implies that $x_{i+1}^{k,i+1} \leq x_{i+1}''$ and hence

$$x^{k,i+1} = (x_{i+1}^{k,i+1}, x_{-(i+1)}^{k,i}) \leq (x_{i+1}'', x_{-(i+1)}'') = x''.$$

By induction, $x^{k,i} \leq x''$ for all k and i and so $\lim_{k\to\infty} x^{k,i} \leq x''$. Therefore, the limit point of $\{x^{k,i}\}$ is the least equilibrium point. □

Example 4.3.2. Consider the noncooperative game $(N, S, \{f_i : i \in N\})$ with $n = 2$, $m_1 = m_2 = 1$, $S = \{(1 - 1/k, 1 - 1/k) : k = 1, 2, \dots\} \cup \{(1 - 1/(k + 1), 1 - 1/k) : k = 1, 2, \dots\} \cup \{(1, 1)\} \cup \{(1, 2)\}$, $f_1(x) = x_1$, and $f_2(x) = x_2$. The unique equilibrium point is $(1, 2)$. For this example, Algorithm 4.3.1 generates $x^{k,0} = (1 - 1/(k + 1), 1 - 1/(k + 1))$ and $x^{k,1} = (1 - 1/(k + 2), 1 - 1/(k + 1))$ for $k = 0, 1, \dots$. However, $\lim_{k\to\infty} x^{k,0} = \lim_{k\to\infty} x^{k,1} = (1, 1) \neq (1, 2)$, so the sequence generated by Algorithm 4.3.1 does not approximate an equilibrium point. This example satisfies the conditions of Theorem 4.2.1 and Theorem 4.3.1, and it satisfies the conditions of Theorem 4.3.2 except for the condition that $S_2(x_{-2})$ be a lower semicontinuous correspondence from S_{-2} into subsets of R^1. To confirm this, let $z^k = 1 - 1/k$ for $k = 1, 2, \dots$ and $z' = 1$ so $\lim_{k\to\infty} z^k = z'$ and 2 is in $S_2(z')$ but $S_2(z^k) = \{1 - 1/k, 1 - 1/(k - 1)\}$ for $k \geq 2$ so there do not exist y^k in $S_2(z^k)$ for $k = 1, 2, \dots$ with $\lim_{k\to\infty} y^k = 2$. Thus $S_2(x)$ is not a lower semicontinuous correspondence on S_{-2}.

Example 4.3.3. Consider the noncooperative game $(N, S, \{f_i : i \in N\})$ with $n = 2$, $m_1 = m_2 = 1$, $S = [-1, 0] \times [-1, 1]$, $f_1(x) = -(2x_1 - x_2)^2$ for x in S, $f_2(x) = -(2x_2 - x_1)^2$ for x in $S \setminus (\{0\} \times [0, 1])$, and $f_2(x) = x_2^2$ for x in $\{0\} \times [0, 1]$. The unique equilibrium point is $(0, 1)$. For this example, Algorithm 4.3.1 generates $x^{0,0} = (-1, -1)$, $x^{k,0} = (-(1/2)^{2k-1}, -(1/2)^{2k})$ for $k = 1, 2, \dots$, and $x^{k,1} = (-(1/2)^{2k+1}, -(1/2)^{2k})$ for $k = 0, 1, \dots$. However, $\lim_{k\to\infty} x^{k,0} = \lim_{k\to\infty} x^{k,1} = (0, 0) \neq (0, 1)$, so the sequence generated by Algorithm 4.3.1 does not approximate an equilibrium point. This example satisfies the conditions of Theorem 4.2.1 and Theorem 4.3.1. (Note that $f_1(x)$ and $f_2(x)$ are differentiable with respect to x_2 on S for each x_1 in $[-1, 0]$ and their derivatives with respect to x_2 are an increasing function of x_1.) Furthermore, this example satisfies the conditions of Theorem 4.3.2 except for the continuity of $f_2(x)$ (which is continuous in x_2 and upper semicontinuous in x, but not continuous in x_1).

The properties shown in Theorem 4.3.1 and Theorem 4.3.2 for supermodular games also hold for quasisupermodular games. Establishing this extension involves observing that part (c) and part (g) of Lemma 4.2.2, as used in Theorem 4.3.1 and Theorem 4.3.2, hold for quasisupermodular games as well as supermodular games.

4.3.2 Simultaneous Optimization

Algorithm 4.3.2 formally describes the procedure for simultaneous optimization by the players.

Algorithm 4.3.2. Given a noncooperative game $(N, S, \{f_i : i \in N\})$, proceed as follows to generate a (finite or infinite) sequence of feasible joint strategies.

(a) If S has a least element $\inf(S)$, set $x^0 = \inf(S)$. Otherwise, stop.
(b) Given x^k in S for any nonnegative integer k, let x^{k+1} be the least element of the best joint response correspondence $Y(x^k)$ if such a joint strategy exists. Otherwise, stop.
(c) Set $k = k + 1$. Return to step (b) and continue.

In the case where each player's set of feasible strategies is independent of the strategies of the other players (that is, $S = \times_{i \in N} S_i$), Algorithm 4.3.2 corresponds to the iterative decision-making process by which each of the n players concurrently and individually chooses its next strategy by maximizing that player's own payoff function under the assumption that the other $n - 1$ players hold their strategies unchanged. A new joint strategy is put together by combining these n individually determined strategies, and the next iteration then begins.

Theorem 4.3.3 establishes monotonicity, stopping, and efficiency properties for Algorithm 4.3.2 when applied to a supermodular game with certain regularity conditions. Part (a) of Theorem 4.3.2 shows that the sequence x^k generated by Algorithm 4.3.2 is increasing in k. This reduces the problem of finding x^{k+1} given x^k in step (b) of Algorithm 4.3.2 from a maximization problem over $S(x^k)$ to a maximization problem over $S(x^k) \cap [x^k, \infty)$. Part (b) of Theorem 4.3.2 shows that if a joint strategy generated by step (b) of Algorithm 4.3.1 is identical with the preceding joint strategy, then this joint strategy is an equilibrium point. This result can be incorporated into a stopping rule whereby step (c) of Algorithm 4.3.2 would terminate the algorithm whenever step (b) leaves the joint strategy unchanged, as this joint strategy must be an equilibrium point. Furthermore, part (c) of Theorem 4.3.3 shows that if Algorithm 4.3.2 ever generates an equilibrium point, then it continues to generate that same equilibrium point at all subsequent iterations of step (b). Hence, a joint strategy generated in step (b) of Algorithm 4.3.2 is an equilibrium point if and only if it is the same as the previous joint strategy generated by Algorithm 4.3.2. Part (d) of Theorem 4.3.3 indicates that Algorithm 4.3.2 is very efficient computationally when S is finite.

Theorem 4.3.3. *Consider a supermodular game* $(N, S, \{f_i : i \in N\})$ *for which the set* S *of feasible joint strategies is nonempty and compact and the payoff function* $f_i(y_i, x_{-i})$ *is upper semicontinuous in* y_i *on* $S_i(x_{-i})$ *for each* x_{-i} *in* S_{-i} *and each* i.
(a) *Algorithm* 4.3.2 *never stops at step* (a) *or step* (b) *and Algorithm* 4.3.2 *generates an infinite sequence* x^k *that is increasing in* k *for* $k = 0, 1, \ldots$. *Hence, there exists a feasible joint strategy* x' *in* S *such that* $\lim_{k \to \infty} x^k = x'$.

(b) *If a feasible joint strategy appears two successive times in the sequence* $\{x^k : k \geq 0\}$ *generated by Algorithm* 4.3.2, *then it is an equilibrium point.*
(c) *If Algorithm* 4.3.2 *generates an equilibrium point at some iteration, then Algorithm* 4.3.2 *generates that same equilibrium point at each subsequent iteration.*
(d) *If, in addition, S has a finite number of elements and q is an upper bound on the number of elements in any chain contained in S, then Algorithm* 4.3.2 *generates an equilibrium point in no more than* $q - 1$ *iterations of step* (b).

Proof. Algorithm 4.3.2 does not stop at step (a) because the nonempty compact sublattice S has a least element by Corollary 2.3.2. Algorithm 4.3.2 does not stop at step (b) by part (d) of Lemma 4.2.2. Therefore, step (a) and step (b) of Algorithm 4.3.2 proceed without stopping, and Algorithm 4.3.2 generates an infinite sequence of feasible joint strategies.

Since x^0 is the least element of S, $x^0 \leq x^1$. Now suppose that k' is a positive integer such that $x^k \leq x^{k+1}$ for $k = 0, \ldots, k' - 1$. Since this supposition holds for $k' = 1$, it suffices to show that $x^{k'} \leq x^{k'+1}$ for the proof of part (a) to follow by induction. Because $x^{k'}$ is the least element of $Y(x^{k'-1})$, $x^{k'+1}$ is the least element of $Y(x^{k'})$, and $x^{k'-1} \leq x^{k'}$ by the induction hypothesis, part (h) of Lemma 4.2.2 implies that $x^{k'} \leq x^{k'+1}$. The proof of part (a) is completed by noting that an increasing sequence in a compact set has a limit point.

If $x^{k'} = x^{k'+1}$ for some k' then $x^{k'} = x^{k'+1}$ is in $Y(x^{k'})$ by the construction of step (b) of Algorithm 4.3.2 and $x^{k'}$ is a fixed point of $Y(x)$. Then $x^{k'}$ is an equilibrium point by Lemma 4.2.1, and part (b) holds.

Suppose that $x^{k'}$ is an equilibrium point for some nonnegative integer k'. To establish part (c), it suffices to show that $x^{k'+1} = x^{k'}$. Because $x^{k'}$ is an equilibrium point, $x^{k'}$ is in $Y(x^{k'})$ by Lemma 4.2.1 and so $x^{k'+1} \leq x^{k'}$ because step (b) of Algorithm 4.3.2 constructs $x^{k'+1}$ as the least element of $Y(x^{k'})$. By part (a), $x^{k'} \leq x^{k'+1}$. Hence, $x^{k'} = x^{k'+1}$.

Now include the additional hypotheses of part (d). By part (a), x^k is increasing in k for $k = 0, 1, \ldots$. Thus $\{x^k : k \geq 0\}$ is a chain and has no more than q distinct elements. Because x^k is increasing and there are no more than q distinct elements in $\{x^k : k \geq 0\}$, Algorithm 4.3.2 must eventually generate some feasible joint strategy such that the same feasible joint strategy is generated at all subsequent iterations of step (b). By part (b), that last distinct feasible joint strategy generated must be an equilibrium point. Since x^k is increasing in k all appearances of a feasible joint strategy in this sequence must be consecutive, and by part (b) and part (c) no feasible joint strategy except the last distinct one generated can appear more than once in the sequence $\{x^k : k \geq 0\}$. Thus Algorithm 4.3.2 must attain an equilibrium point in no more than $q - 1$ iterations of step (b), completing the proof of part (d). □

Consider efficiency implications of the bound in part (d) of Theorem 4.3.3. If S is finite, $S_i(x_{-i}) = S_i$ for each x_{-i} in S_{-i} and each i, and S_i has q_i elements for $i = 1, \ldots, n$, then adding one element to $S_{i'}$ for some player i' adds $\prod_{i \neq i'} q_i$ elements to S but the bound on the number of iterations required by Algorithm 4.3.2 increases by either 0 or 1. If S is finite and q'_i is an upper bound on the number of elements in any chain contained in S_i, then $\sum_{i=1}^{n}(q'_i - 1) + 1$ is an upper bound on the number of elements in any chain contained in S and so $\sum_{i=1}^{n}(q'_i - 1)$ is an upper bound on the number of iterations of step (b) required for Algorithm 4.3.2 to attain an equilibrium point by part (d) of Theorem 4.3.3. This bound is proportional to the number of players when q'_i does not depend on i. If $a_j \leq b_j$ are integers for $j = 1, \ldots, m$ and S is a subset of $\{z : a_j \leq z_j \leq b_j$ and z_j integer for $1 \leq j \leq m\}$, then $\sum_{j=1}^{m}(b_j - a_j) + 1$ is an upper bound on the number of elements in any chain contained in S while S may have as many as $\prod_{j=1}^{m}(b_j - a_j + 1)$ elements. Example 4.3.4, using the same game as in Example 4.3.1, further illustrates the efficiency of Algorithm 4.3.2 for S finite.

Example 4.3.4. Consider a supermodular game $(N, S, \{f_i : i \in N\})$ with $n = 20$, $S_i = \{1, \ldots, 10\}$ for each i, and $S = \times_{i=1}^{n} S_i$. Let $q = \sum_{i=1}^{n}(|S_i| - 1) + 1 = (20)(9) + 1 = 181$, so q is an upper bound on the number of elements in any chain contained in S. The set S contains $\prod_{i=1}^{n} |S_i| = 10^{20}$ feasible joint strategies. However, Algorithm 4.3.2 generates an equilibrium point in no more than $q - 1 = 180$ iterations of step (b) by part (d) of Theorem 4.3.3. Since each iteration of step (b) in Algorithm 4.3.2 requires looking at no more than $q = (20)(9) + 1 = 181$ joint strategies and making at most $q - 1 = (20)(9) = 180$ comparisons, one can find an equilibrium point for this noncooperative game with 10^{20} feasible joint strategies by looking at no more than $(180)(181) = 32,580$ joint strategies and making no more than $(180)(180) = 32,400$ comparisons.

Theorem 4.3.4 gives conditions for the sequence generated by Algorithm 4.3.2 to converge to an equilibrium point. Theorem 4.3.4 constructively establishes the existence of an equilibrium point for supermodular games (with certain regularity conditions), while the existence proof of Theorem 4.2.1 is not constructive. The hypotheses of Theorem 4.3.4 are stronger than those of Theorem 4.2.1 (and Theorem 4.3.3) by additionally requiring that $S(x)$ is a lower semicontinuous correspondence and that each $f_i(x)$ is continuous in x on S (rather than being upper semicontinuous in x_i for each x_{-i}). Example 4.3.5 and Example 4.3.6 show that these additional assumptions cannot be dispensed with entirely.

Theorem 4.3.4. *Consider a supermodular game $(N, S, \{f_i : i \in N\})$ for which the set S of feasible joint strategies is nonempty and compact, $S(x)$ is a lower semicontinuous correspondence from S into subsets of R^m, and the payoff function $f_i(x)$ is continuous in x on S for each i. The limit point of the increasing sequence generated by Algorithm* 4.3.2 *is an equilibrium point. Furthermore, it is the least equilibrium point.*

Proof. Let $\{x^k : k \geq 0\}$ be the sequence generated by Algorithm 4.3.2. By part (a) of Theorem 4.3.3, this sequence is increasing in k and converges to a limit point x'. Pick any y' in $S(x')$. Since $\lim_{k\to\infty} x^k = x'$ and $S(x)$ is a lower semicontinuous correspondence, there exist y^k in $S(x^k)$ for $k = 0, 1, \ldots$ such that $\lim_{k\to\infty} y^k = y'$. Because x^{k+1} is in $Y(x^k) = \text{argmax}_{y\in S(x)} g(y, x^k)$ by the construction of step (b) of Algorithm 4.3.2 and because y^k is in $S(x^k)$, $g(y^k, x^k) \leq g(x^{k+1}, x^k)$ for each k. Then by the continuity of $f_i(x)$ in x for each i,

$$g(y', x') = \lim_{k\to\infty} g(y^k, x^k) \leq \lim_{k\to\infty} g(x^{k+1}, x^k) = g(x', x').$$

Because $g(y', x') \leq g(x', x')$ for each y' in $S(x')$, x' is a fixed point of $Y(x)$ and so x' is an equilibrium point by Lemma 4.2.1.

Let x' be the limit point of the sequence generated by Algorithm 4.3.2 and let x'' be any equilibrium point. Since x^0 is the least element of S by step (a) of Algorithm 4.3.2, $x^0 \leq x''$. Now suppose that $x^k \leq x''$ for some nonnegative integer k. Since x^{k+1} is the least element of $Y(x^k)$ by the construction of step (b) of Algorithm 4.3.2, x'' is in $Y(x'')$ by Lemma 4.2.1, and $x^k \leq x''$, part (h) of Lemma 4.2.2 implies that $x^{k+1} \leq x''$. By induction, $x^k \leq x''$ for each k and so $x' = \lim_{k\to\infty} x^k \leq x''$. □

Example 4.3.5. Consider the noncooperative game given in Example 4.3.2. Algorithm 4.3.2 generates $x^k = (1 - 1/(k/2 + 1), 1 - 1/(k/2 + 1))$ for k even and $x^k = (1 - 1/((k-1)/2 + 2), 1 - 1/((k-1)/2 + 1))$ for k odd. However, the unique equilibrium point is $(1, 2)$ and $\lim_{k\to\infty} x^k = (1, 1) \neq (1, 2)$, so the sequence generated by Algorithm 4.3.2 does not approximate an equilibrium point. Because $S_2(x_{-2})$ is not a lower semicontinuous correspondence (as shown in Example 4.3.2), $S(x)$ is not a lower semicontinuous correspondence on S. This example satisfies the conditions of Theorem 4.2.1 and Theorem 4.3.3, and it satisfies the conditions of Theorem 4.3.4 except for the lower semicontinuity of the correspondence $S(x)$.

Example 4.3.6. Consider the noncooperative game given in Example 4.3.3. Algorithm 4.3.2 generates $x^k = (-(1/2)^k, -(1/2)^k)$ for each k. However, the unique equilibrium point is $(0, 1)$ and $\lim_{k\to\infty} x^k = (0, 0) \neq (0, 1)$, so the sequence generated by Algorithm 4.3.2 does not converge to an equilibrium point. This example satisfies the conditions of Theorem 4.2.1 and Theorem 4.3.3, and it satisfies the conditions of Theorem 4.3.4 except for the continuity of $f_2(x)$ (which is continuous in x_2 and upper semicontinuous in x, but not continuous in x_1).

If step (a) is modified so that Algorithm 4.3.1 and Algorithm 4.3.2 begin with the greatest instead of the least element of S and "least" is replaced by "greatest" in step (b) of these algorithms, then by dual arguments all results in this section would hold with "increasing" replaced by "decreasing" and "least" replaced by "greatest" and so each algorithm would generate a sequence of feasible joint strategies converging downwards to the greatest equilibrium point.

Both Algorithm 4.3.1 and Algorithm 4.3.2 could be modified so that at each iteration the solution chosen for the maximization problem may be any optimal solution (not necessarily the least one) that assures that the sequence generated is increasing.

If $S = \times_{i \in N} S_i$, then the properties shown in Theorem 4.3.3 and Theorem 4.3.4 for supermodular games with finitely many players also hold for quasisupermodular games with infinitely many players. Establishing this extension involves observing that part (d) and part (h) of Lemma 4.2.2, as used in Theorem 4.3.3 and Theorem 4.3.4, hold for quasisupermodular games with infinitely many players if $S = \times_{i \in N} S_i$ and making suitable modifications in the first paragraph of the proof of Theorem 4.3.4.

4.4 Examples of Supermodular Games

This section gives eight examples of supermodular games, so the theory of Section 4.2 and the properties established for the algorithms in Section 4.3 apply to each of these examples. Subsection 4.4.1 models competitive pricing for firms producing and marketing substitute products. Subsection 4.4.2 involves non-price decisions for firms producing and marketing complementary products. Subsection 4.4.3 considers a multimarket oligopoly where multiple firms are engaged in activities in multiple markets. Subsection 4.4.4 considers the strategies of countries engaged in an arms race, with each country determining its arms level. Subsection 4.4.5 models firms engaged in a search for prospective trading partners, where each firm must determine the level of effort put into its search. Subsection 4.4.6 models optimal consumption of multiple products by multiple consumers, where each consumer of each product benefits when a greater total amount of each product is consumed by all consumers. Subsection 4.4.7 models the location of multiple facilities by multiple firms, where the cost to each firm depends on the distances of its new facilities from each other, from new facilities of the other firms, and from already existing facilities. Subsection 4.4.8 is a noncooperative game version of the minimum cut problem, where each player determines a subset of the nodes in a cut. Further examples of supermodular games are noted in Fudenberg and Tirole [1991], Milgrom and Roberts [1990a, 1991], Milgrom and Shannon [1994], and Vives [1990].

4.4.1 Pricing Game with Substitute Products

Consider a set $N = \{1, \dots, n\}$ of firms (players), where each firm i produces a single product i and the n products are substitutes for one another in a sense made more precise below. Each firm i sets the price x_i of its product i. The vector of these n prices is $x = (x_1, \dots, x_n)$. The price vector x must be contained in a subset S of R^n. The demand for product i depends on the price vector x according to a known demand function $D_i(x)$. Each firm i produces $D_i(x)$ units of product i to satisfy demand. There is a unit production cost t_i for product i, so the total production cost for product i is $t_i D_i(x)$. The revenue for product i given x is $x_i D_i(x)$. Assume that S is a subset of $\times_{i \in N}[t_i, \infty)$, so no feasible price is below the product's production cost. The net profit for firm i is the payoff function

$$f_i(x) = (x_i - t_i) D_i(x).$$

Given the prices for all firms other than i, firm i chooses x_i to maximize $f_i(x)$ over feasible x_i. This is an n-player noncooperative game. In the case for which $S = R^n$ and each $D_i(x)$ is an affine function, Levitan and Shubik [1971] consider a version of the above model and find a closed-form algebraic expression for an equilibrium point.

Assume that S is a nonempty compact sublattice of R^n and that, for all distinct firms i' and i'', $D_{i'}(x)$ is increasing in $x_{i''}$ and has increasing differences in $(x_{i'}, x_{i''})$. By part (b) of Example 2.2.7, the assumption that S is a sublattice of R^n permits the form of S to reflect upper and lower bounds on each price as well as upper and lower bounds on the difference between the prices of any pair of distinct products. The assumptions on the demand functions imply that $f_i(y_i, x_{-i})$ has increasing differences in (y_i, x_{-i}) for each i. Then $(N, S, \{f_i : i \in N\})$ is a supermodular game. The assumption that $D_{i'}(x)$ increases with $x_{i''}$ for all distinct firms i' and i'' states that the demand for any product increases with the price of any other product, and this is a standard condition for substitute products. The assumption that $D_{i'}(x)$ has increasing differences in $(x_{i'}, x_{i''})$ for all distinct firms i' and i'' means that decreasing the price of any product results in a greater increase in the demand for that product for lower levels of the price of any other product; that is, the demand for any product is more sensitive to its price when any other product is more competitive by virtue of its lower price. (The discussion of a price decrease for a product leading to an increase in the demand for that product is only for purposes of illustration. The present conditions do not preclude the possibility that demand for a product could increase with its price.) The assumption of increasing differences would hold if $D_i(x)$ is a separable function of x. Topkis [1979] formulates the above model and observes that the game is a supermodular game under the present assumptions.

Theorem 4.2.3 implies that the price of each firm in the greatest (least) equilibrium point decreases as additional firms enter the market. (Establishing this conclusion based on Theorem 4.2.3 involves including for each firm a feasible price so large that with this price a firm would not receive any demand for its product and the demands for the products of the other firms would be unaffected by the presence of that firm in the market. Furthermore, the ordering relation on the set of prices should be the dual of the usual partial ordering $\leq$ on R^1.)

Now assume that S is a subset of $\times_{i\in N}(t_i, \infty)$ (instead of $\times_{i\in N}[t_i, \infty)$ as assumed above) and assume that, for all distinct firms i' and i'', $D_{i'}(x)$ is positive and $\log(D_{i'}(x))$ has increasing differences in $(x_{i'}, x_{i''})$ (instead of $D_{i'}(x)$ being increasing in $x_{i''}$ and having increasing differences in $(x_{i'}, x_{i''})$ as assumed above). When each demand function $D_i(x)$ is differentiable in i, the latter assumption is equivalent to the property that each firm's price elasticity of demand is a decreasing function of the prices of the other firms' products. (See Example 2.6.11.) Because each firm's payoff function can, in effect, be replaced by any strictly increasing function of its payoff function, the payoff function of each firm i can be taken as

$$g_i(x) = \log(f_i(x)) = \log((x_i - t_i)D_i(x)) = \log(x_i - t_i) + \log(D_i(x)).$$

Let $t = (t_1, \ldots, t_n)$ and denote the dependence of $g_i(x)$ on t by $g_i^t(x)$. The noncooperative game $(N, S, \{g_i^t : i \in N\})$ with this transformed payoff function is a supermodular game for each t by hypothesis, Theorem 2.6.4, and part (b) of Lemma 2.6.1. Furthermore, each $g_i^t(x)$ has increasing differences in (x_i, t) by part (b) of Lemma 2.6.2 and Theorem 2.6.1. By Theorem 4.2.2, the greatest (least) equilibrium point for each unit production cost vector t is increasing with t. Milgrom and Roberts [1990a] use the log transformation to establish the monotonicity with respect to t of equilibrium points in this model.

4.4.2 Production Game with Complementary Products

The model of this subsection extends a version of the decision model of Example 3.3.2 to a noncooperative game with multiple firms. Consider a set of firms (players) $N = \{1, \ldots, n\}$, where each firm produces multiple products in a single period. The model aggregates properties of the products produced by each firm, so that each firm is modeled as producing a single product. The time period is τ. The price p_i of the (aggregated) product of firm i is determined by the market independent of any decisions of the n firms. Firm i produces η_i products, which are modeled as a single aggregated product except for their number denoted as η_i. The quality of the product of firm i is q_i. The level of technology used in the manufacturing process of firm i is θ_i. Firm i spends α_i on advertising for its product. The product of firm i is exposed to a market of size σ_i. The strategy of firm i is $x_i = (\eta_i, q_i, \theta_i, \alpha_i, \sigma_i)$, which must be selected from a nonempty subset S_i of R^5. The joint strategy of all n firms is $x = (x_1, \ldots, x_n)$, and the set of feasible joint strategies is $\times_{i=1}^n S_i$. The demand for the product of firm i in period τ is $\mu_i(x, \tau)$. Each firm i produces $\mu_i(x, \tau)$ units of product i in period τ to satisfy demand. The cost of producing z units of the product of firm i is $c_i(z, x_i, \tau)$ (which depends on the components η_i, q_i, and θ_i of x_i but not on the components α_i and σ_i). The cost to firm i associated with its technology-related variables and independent of the level of production is $\kappa_i(x_i, \tau)$ (which depends on the components η_i, q_i, and θ_i of x_i but not on the components α_i and σ_i). In addition to its advertising cost α_i, there is a marketing cost $\psi_i(\sigma_i, \tau)$ to firm i for servicing a market of size σ_i in time period τ. The payoff function for each firm i is

$$f_i(x, \tau) = p_i \mu_i(x, \tau) - c_i(\mu_i(x, \tau), x_i, \tau) - \kappa_i(x_i, \tau) - \alpha_i - \psi_i(\sigma_i, \tau).$$

Suppose that each S_i is a sublattice; each $\mu_i(x, \tau) = \mu_i(x_i, x_{-i}, \tau)$ is increasing in x, is supermodular in x_i, and has increasing differences in (x_i, x_{-i}); each $c_i(z, x_i, \tau)$ is concave in z and submodular in (z, x_i); each $p_i z - c_i(z, x_i, \tau)$ is increasing in z; and each $\kappa_i(x_i, \tau)$ is submodular in x_i. Then $(N, S, \{f_i : i \in N\})$ is a supermodular game by Lemma 2.6.4 and part (b) of Lemma 2.6.1.

Theorem 4.2.3 can be used to give conditions for the greatest (least) equilibrium point to increase as additional firms enter the market.

Now consider the time period τ as a parameter. In addition to the assumptions of the first two paragraphs of this subsection, suppose that each S_i is compact; each $\mu_i(x, \tau)$ is upper semicontinuous in x, is increasing in τ, and has increasing differences in (x_i, τ); each $c_i(z, x_i, \tau)$ is lower semicontinuous in (z, x_i) and has decreasing differences in $((z, x_i), \tau)$; each $\kappa_i(x_i, \tau)$ is lower semicontinuous in x_i and has decreasing differences in (x_i, τ); and each $\psi_i(\sigma_i, \tau)$ is lower semicontinuous in σ_i and has decreasing differences in (σ_i, τ). By Lemma 2.6.4, Theorem 2.6.2, and Theorem 2.6.1, $f_i(x, \tau) = f_i(x_i, x_{-i}, \tau)$ has increasing differences in (x_i, τ) for each i. By Theorem 4.2.2, the greatest (least) equilibrium point for each time period τ exists and is increasing in τ.

4.4.3 Multimarket Oligopoly

Consider a set of firms (players) $N = \{1, \ldots, n\}$ engaged in activities in each of m markets $j = 1, \ldots, m$. The strategy of firm i in market j is a $k_{i,j}$-vector $x_{i,j}$, where $x_{i,j}$ is the vector of activity levels of firm i in market j, $k_{i,j}$ is a positive integer, and $x_{i,j}$ must be in a nonempty subset $S_{i,j}$ of $R^{k_{i,j}}$. The strategy of firm i is $x_i = (x_{i,1}, \ldots, x_{i,m})$, the joint strategy of all n firms is $x = (x_i, \ldots, x_n)$, and the set of feasible joint strategies is $S = \times_{i \in N}(\times_{j=1}^{m} S_{i,j})$. The return to firm i from its activities in market j is a function $h_{i,j}(x_{1,j}, \ldots, x_{n,j})$ of the strategies of all n firms in market j. There is a cost $g_i(x_i)$ to each firm i depending on the strategies of firm i in all m markets. (The following can be easily generalized to the case where this cost depends on all components of the joint strategy x rather than just x_i.) The payoff function for each firm i is

$$f_i(x) = \textstyle\sum_{j=1}^{m} h_{i,j}(x_{1,j}, \ldots, x_{n,j}) - g_i(x_i).$$

The present model generalizes a model of Bulow, Geanakoplos, and Klemperer [1985] having two firms ($n = 2$), two markets ($m = 2$), one firm being a monopoly in one market, the firms being a duopoly in the other market, and each $k_{i,j} = 1$.

If each $S_{i,j}$ is a sublattice, each $h_{i,j}(x_{1,j}, \ldots, x_{n,j})$ is supermodular in $x_{i,j}$, each $h_{i,j}(x_{1,j}, \ldots, x_{n,j})$ has increasing differences in $(x_{i,j}, x_{i',j})$ for each $i' \neq i$, and each $g_i(x_i)$ is submodular in x_i, then $(N, S, \{f_i : i \in N\})$ is a supermodular game. By Theorem 2.6.1, these hypotheses mean that the components of the strategy of each firm for each market are complementary (with respect to the return function for that firm in that market), each component of the strategy of each firm for each market is complementary (with respect to the return function for that firm in that market) with each component of the strategy of each other firm for that market, and all components of the strategy of each firm are complementary (with respect to that firm's cost function).

Theorem 4.2.3 can be used to give conditions for the activity levels of each firm in each market in the greatest (least) equilibrium point to increase as additional firms enter the markets.

Now suppose that each return function and each cost function depend on a parameter t included in a subset T of R^q, with these functions denoted

$h^t_{i,j}(x_{1,j}, \ldots, x_{n,j})$ and $g^t_i(x_i)$. The payoff function for each firm i takes the form

$$f^t_i(x) = \sum_{j=1}^m h^t_{i,j}(x_{1,j}, \ldots, x_{n,j}) - g^t_i(x_i).$$

In addition to the hypotheses of the first two paragraphs of this subsection, assume that each $S_{i,j}$ is compact, each $h^t_{i,j}(x_{1,j}, \ldots, x_{n,j})$ is upper semicontinuous in $x_{i,j}$, each $g^t_i(x_i)$ is lower semicontinuous in x_i, each $h^t_{i,j}(x_{1,j}, \ldots, x_{n,j})$ has increasing differences in $(x_{i,j}, t)$, and each $g^t_i(x_i)$ has decreasing differences in (x_i, t). The latter two additional assumptions mean that each component of the strategy of each firm for each market is complementary (with respect to the return function for that firm in that market) with each component of the parameter and each component of the strategy of each firm is complementary (with respect to that firm's cost function) with each component of the parameter. By Theorem 4.2.2, there exists a greatest (least) equilibrium point for each parameter t such that this equilibrium point increases with t.

4.4.4 Arms Race Game

Consider a set of countries (players) $N = \{1, \ldots, n\}$ engaged in an arms race. Each country i chooses its arms level x_i from a nonempty subset S_i of R^1. The joint strategy of all n countries is $x = (x_1, \ldots, x_n)$ and the set of feasible joint strategies is $S = \times_{i \in N} S_i$. The payoff function for country i is $f_i(x)$, where $f_i(y_i, x_{-i})$ has increasing differences in (y_i, x_{-i}). This increasing differences hypothesis means that the (perceived) value of any additional arms to any country i increases with the arms levels of all other countries; that is, additional arms are deemed more valuable when the military capabilities of one's adversaries are greater. Then $(N, S, \{f_i : i \in N\})$ is a supermodular game. If each S_i is compact and each $f_i(y_i, x_{-i})$ is upper semicontinuous in y_i, then an equilibrium point exists by Theorem 4.2.1. Indeed, Theorem 4.2.1 implies that there is a least equilibrium point, and a tantalizing question involves how to get the n countries to come to that particular equilibrium point. Theorem 4.3.2 and Theorem 4.3.4 indicate that Algorithm 4.3.1 and Algorithm 4.3.2 provide mechanisms for converging to the least equilibrium point. The present model generalizes a model of Milgrom and Roberts [1990a] for the case $n = 2$ and the payoff function taking the form $f_i(x) = g_i(x_i - x_{-i}) - c_i(x_i)$ where $g(z)$ is concave. (See part (b) of Lemma 2.6.2.) Milgrom and Roberts [1990a, 1994] analyze a dynamic generalization of that model, where the arms depreciate and new investments in arms are made over infinitely many time periods.

Theorem 4.2.3 can be used to give conditions for the arms level of each country in the greatest (least) equilibrium point to increase as additional countries enter the arms race.

Now suppose that each payoff function $f_i(x)$ depends on a parameter t, as $f_i^t(x)$, where the parameter t is contained in a partially ordered set T. In addition to the hypotheses of the first paragraph of this subsection, assume that each $f_i^t(y_i, x_{-i})$ has increasing differences in (y_i, t); that is, the additional value of additional arms for any country i is greater for higher values of the parameter t. By Theorem 4.2.2, the level of arms for each country in the greatest (least) equilibrium point for each t increases with t.

4.4.5 Trading Partner Search Game

Consider an economy composed of a set of firms (players) $N = \{1, \ldots, n\}$, with each firm engaged in a search for prospective trading partners. The level of search determined by each firm i is denoted x_i, where x_i must be selected from a nonempty subset S_i of R^1. The joint strategy for all n firms is $x = (x_1, \ldots, x_n)$ and the set of feasible joint strategies is $S = \times_{i \in N} S_i$. The cost to firm i for exerting a level of search x_i is $c_i(x_i)$. Given a joint strategy x, the expected return from trade for firm i is $g_i(x)$. The payoff function for each firm i is

$$f_i(x) = g_i(x) - c_i(x_i).$$

If each $g_i(x)$ has increasing differences in (x_i, x_{-i}), then $(N, S, \{f_i : i \in N\})$ is a supermodular game. The hypothesis that $g_i(x)$ has increasing differences in (x_i, x_{-i}) means that any additional effort exerted by firm i in its search for trading partners is (expected to be) more profitable for that firm when other firms exert higher levels of effort in their search for trading partners. The present search model generalizes a model of Milgrom and Roberts [1990a] that considers the symmetric payoff function $f_i(x) = \alpha x_i(\sum_{k \neq i} x_k) - c(x_i)$ for each player i.

Theorem 4.2.3 can be used to give conditions for the level of search by each firm in the greatest (least) equilibrium point to increase as additional firms enter the economy and become prospective trading partners.

Now suppose, in addition to the assumptions of the first paragraph of this subsection, that each $c_i(x_i)$ and $g_i(x)$ also depends on a parameter t for t contained in a partially ordered set T and this dependence is denoted $c_i^t(x_i)$ and $g_i^t(x)$. The parameterized payoff function for firm i is

$$f_i^t(x) = g_i^t(x) - c_i^t(x_i).$$

Assume also that each S_i is compact, each $c_i^t(x_i)$ is lower semicontinuous in x_i, each $g_i^t(x)$ is upper semicontinuous in x_i, each $c_i^t(x_i)$ has decreasing differences in (x_i, t), and each $g_i^t(x)$ has increasing differences in (x_i, t). With respect to both the search cost $c_i^t(x_i)$ and the expected return function $g_i^t(x)$, the parameter t is complementary with the search effort x_i. The payoff function $f_i(x, t)$ has increasing differences in (x_i, t). By Theorem 4.2.2, the greatest (least) equilibrium point for each t is increasing in t.

4.4.6 Optimal Consumption Game with Multiple Products

Suppose that there are m products $j = 1, \ldots, m$ and each consumer (player) i in the set $N = \{1, \ldots, n\}$ chooses a subset of these m products to consume. The strategy of consumer i is denoted by a m-vector $x^i = (x^i_1, \ldots, x^i_m)$ such that $x^i_j = 1$ if consumer i chooses to consume product j and $x^i_j = 0$ if consumer i chooses not to consume product j. Consumer i may only choose certain combinations of products, so x^i is restricted to a subset S_i of $\times_{j=1}^m \{0, 1\}$. The joint strategy is $x = (x^1, \ldots, x^n)$ and the set of feasible joint strategies is $S = \times_{i \in N} S_i$. If consumer i chooses to consume product j, then i consumes a fixed amount $a_{i,j} > 0$ of product j. If consumer i consumes product j, then i incurs a cost that depends on the total amount of product j consumed by all of the n consumers. That cost is $c_{i,j}(\sum_{k=1}^n a_{k,j} x^k_j)$. Consumer i incurs no cost with respect to product j if i chooses not to consume product j. Define $b_{i,j}(x^i_j, z) = c_{i,j}(z)$ if $x^i_j = 1$ and $b_{i,j}(x^i_j, z) = 0$ if $x^i_j = 0$. The cost to consumer i for product j is $b_{i,j}(x^i_j, \sum_{k=1}^n a_{k,j} x^k_j)$. In consuming the products indicated by a strategy x^i, consumer i receives a return $r_i(x^i)$. The payoff function for consumer i given joint strategy x in S is

$$f_i(x) = r_i(x^i) - \sum_{j=1}^m b_{i,j}\big(x^i_j, \sum_{k=1}^n a_{k,j} x^k_j\big).$$

The present model is from Topkis [1979].

If each S_i is a sublattice of R^m, each $c_{i,j}(z)$ is decreasing on $[a_{i,j}, \infty)$, and each $r_i(x^i)$ is supermodular on S_i, then $(N, S, \{f_i : i \in N\})$ is a supermodular game. The property that $c_{i,j}(z)$ is decreasing implies that $b_{i,j}(x^i_j, \sum_{k=1}^n a_{k,j} x^k_j)$ has decreasing differences in $(x^i_j, x^{i'}_j)$ on $\{0, 1\} \times \{0, 1\}$ for all i, j, and $i' \neq i$. The condition that $r_i(x^i)$ is supermodular indicates that the m products are complementary from the point of view of (the return function of) consumer i.

The set S_i is a sublattice of R^m if $S_i = \times_{j=1}^m \{0, 1\}$. By part (c) of Example 2.2.7, the set S_i is also a sublattice if it includes constraints that consumer i can choose certain products only if i chooses certain other products; the inequality $x^i_{j'} - x^i_{j''} \leq 0$ defines a sublattice, and that inequality requires that consumer i can choose product j' only if i also chooses product j''.

Suppose that $d_j(z)$ is the cost of producing z units of any product j for the n consumers, where $d_j(0) = 0$ and $d_j(z)$ is concave on $[0, \infty)$. There are increasing returns to scale in the production of each product j. Suppose that the cost of producing each product j is allocated to the n consumers in proportion to their use of product j. The cost to consumer i as a result of a decision to consume product j is $c_{i,j}(z) = (a_{i,j}/z)d_j(z)$, where a total of $z > 0$ units of product j are consumed by the n consumers. The properties that $d_j(0) = 0$ and that $d_j(z)$ is concave on $[0, \infty)$ imply that $d_j(z)/z$ is decreasing in z on $(0, \infty)$ and therefore $c_{i,j}(z)$ is decreasing in z on $[a_{i,j}, \infty)$.

Theorem 4.2.3 can be used to give conditions such that the consumption level of each consumer for each product in the greatest (least) equilibrium point increases as additional consumers enter the market.

4.4.7 Facility Location Game

Consider a facility location problem where each of a set of firms (players) $N = \{1, \ldots, n\}$ chooses locations for disjoint collections of facilities. Each firm i controls the location of m_i new facilities. The new facilities of all firms are denoted $j = 1, \ldots, m$, where $m = \sum_{i=1}^{n} m_i$. The set Q_i consists of the m_i new facilities located by firm i. Firm i chooses the location x^j of each new facility j in Q_i from a set S_j, where x^j is a 2-vector and S_j is a subset of R^2. The set of feasible joint strategies is $S = \times_{i \in N}(\times_{j \in Q_i} S_j)$. There are already p existing facilities denoted $k = 1, \ldots, p$, and the 2-vector t^k indicates the location of existing facility k. The function $d(z', z'')$ is a measure of the distance between any two points z' and z'' in R^2. For each ordered pair of new facilities j' and j'' with firm i locating j' and some firm (which may or may not be i) locating j'', there is a cost $c_{j',j''}d(x^{j'}, x^{j''})$ to firm i where $c_{j',j''}$ is nonnegative. For each new facility j located by firm i and each existing facility k, there is a cost $b_{j,k}d(x^j, t^k)$ to firm i where $b_{j,k}$ is nonnegative. (The coefficient $c_{j',j''}$ may, for example, represent the product of the amount of goods shipped from new facility j' to new facility j'' times the cost per unit shipped per unit distance between j' and j'', with $b_{j,k}$ having a similar interpretation.) The strategy for each firm i takes the form $\{x^j : j \in Q_i, x^j \in S_j\}$. Each firm i wants to choose feasible locations for facilities Q_i to minimize its total related costs. That is, each firm i wants to maximize its payoff function

$$f_i(x) = -\sum_{j' \in Q_i} \sum_{j''=1}^{m} c_{j',j''} d(x^{j'}, x^{j''}) - \sum_{j \in Q_i} \sum_{k=1}^{p} b_{j,k} d(x^j, t^k)$$

subject to x^j in S_j for all j in Q_i where $x = (x^1, \ldots, x^m)$. This model is from Topkis [1979].

If each S_j is a nonempty sublattice of R^2 and if $d(z', z'')$ is submodular on R^4, then $(N, S, \{f_i : i \in N\})$ is a supermodular game by part (a) and part (b) of Lemma 2.6.1 and Theorem 2.6.1.

If $d(z', z'') = |z'_1 - z''_1| + |z'_2 - z''_2|$ for z' and z'' in R^2, then $d(z', z'')$ is submodular on R^4 as in part (g) of Example 2.6.2. (This distance measure would seem appropriate where travel between facilities must be on perpendicular city streets or along perpendicular aisles in a machine shop.) Picard and Ratliff [1978] and others cited therein consider the facility location problem corresponding to the case $n = 1$ with this distance measure and each $S_j = R^2$, and they give algorithms that solve that problem.

The distance measure $d(z', z'') = (z'_1 - z''_1)^2 + (z'_2 - z''_2)^2$ is submodular on R^4 as in part (h) of Example 2.6.2. White [1971] considers and solves the facility location problem corresponding to the case $n = 1$ with this distance measure and each $S_j = R^2$. (As in part (h) of Example 2.6.2, the Euclidean distance measure $((z'_1 - z''_1)^2 + (z'_2 - z''_2)^2)^{1/2}$ is not submodular on R^4.)

Let $t = (t^1, \ldots, t^p)$, and now treat the vector t of locations of the p existing facilities as a parameter with the dependence of the payoff function $f_i(x)$ for each player i on the parameter t denoted $f_i^t(x)$. In addition to the assumptions of the first two paragraphs of this subsection, assume that each S_j is compact and that $d(z', z'')$ is lower semicontinuous in (z', z''). Using Theorem 2.6.1, Theorem 4.2.2 implies that there exists a greatest (least) equilibrium point for each t in R^{2p} and this greatest (least) equilibrium point increases with t.

4.4.8 Minimum Cut Game

Consider the following minimum cut game, based on an example in Topkis [1979]. (See Subsection 3.7.1.) There is a directed network with a source s, a sink t, and a set M of m other nodes. The set M is partitioned into n disjoint sets of nodes $M_1, M_2, \ldots, M_n$, where M_i has m_i elements. Thus $M = \cup_{i=1}^n M_i$, $M_{i'} \cap M_{i''} = \emptyset$ for all distinct i' and i'', $\sum_{i=1}^n m_i = m$, and the network has $m+2$ nodes. If X is a subset of M, then $X \cup \{s\}$ is a cut. There is a directed edge joining each ordered pair of nodes. There is a set of players $N = \{1, \ldots, n\}$. Each player i chooses a set of nodes X_i (the strategy of player i) from M_i for the cut $(\cup_{i=1}^n X_i) \cup \{s\}$. The set of feasible joint strategies is $S = \mathcal{P}(M)$. Each player i has a nonnegative real-valued capacity function $c_i(x', x'')$ defined on the edge from any node x' to any node x'', and the associated cut capacity function for player i is

$$g_i(X) = \textstyle\sum_{x' \in X \cup \{s\}} \sum_{x'' \in (M \setminus X) \cup \{t\}} c_i(x', x'')$$

where X is a subset of M and the corresponding cut is $X \cup \{s\}$. The payoff function for player i is

$$f_i(X) = -g_i(X),$$

where the joint strategy X is a subset of M. Player i wants to maximize $f_i(X_i \cup (\cup_{k \neq i} X_k))$ (equivalently, minimize its cut capacity $g_i(X_i \cup (\cup_{k \neq i} X_k))$) over subsets X_i of M_i where X_k is a given subset of M_k for each $k \neq i$. It follows directly from Lemma 3.7.1 that $f_i(X_i \cup (\cup_{k \neq i} X_k))$ is supermodular in X_i for X_i being a subset of M_i and, by Theorem 2.6.1, has increasing differences in $(X_{i'}, X_{i''})$ for all distinct i' and i'', $X_{i'}$ being a subset of $M_{i'}$, and $X_{i''}$ being a subset of $M_{i''}$. Thus $(N, S, \{f_i : i \in N\})$ is a supermodular game.

Maximizing $f_i(X_i \cup (\cup_{k \neq i} X_k))$ over subsets X_i of M_i for given $\cup_{k \neq i} X_k$ is equivalent to finding a minimum cut in a capacitated network with $m_i + 2$ nodes consisting of a source s', a sink t', and the nodes M_i, where the capacity function $c'_i(x', x'')$ is such that $c'_i(x', x'') = c_i(x', x'')$ if x' and x'' are in M_i, $c'_i(s', x'') = c_i(s, x'') + \sum_{x' \in \cup_{k \neq i} X_k} c_i(x', x'')$ for x'' in M_i, $c'_i(x', t') = c_i(x', t) + \sum_{x'' \in \cup_{k \neq i}(M_k \setminus X_k)} c_i(x', x'')$ for x' in M_i, and $c'_i(x', x'') = 0$ otherwise.

An example of the minimum cut game is a noncooperative game version of the selection problem. (See Subsection 3.7.3.) Consider a game with a set of firms (players) $N = \{1, \ldots, n\}$, where each player i chooses whether or not to participate in each of the m_i activities contained in the set M_i. The sets M_i for $i = 1, \ldots, n$ are disjoint. Let $M = \cup_{i=1}^n M_i$, so M has $m = \sum_{i=1}^n m_i$ elements. The set of all activities M is partitioned into two disjoint sets S and T with $S \cup T = M$ and $S \cap T = \emptyset$. The activities of S are not in themselves profitable, so if player i chooses activity x in $M_i \cap S$ then i incurs a cost $c_x \geq 0$. The activities of T are in themselves profitable, so if player i chooses activity x in $M_i \cap T$ then i receives a return $r_x > 0$. Each activity available to a player is complementary with each activity available to that player or to any other player. (Such complementarity could arise where an activity of one player may benefit from the use of transportation facilities, distribution outlets, or product inputs that become available as a result of other activities undertaken by the same player or other players.) The effect of this complementarity between activities is that each player i incurs an additional nonnegative cost $b(x', x'')$ for all x' in M_i and x'' in M if i chooses activity x' and if no player chooses activity x''. This cost structure might induce a player i to choose an unprofitable activity x' from $M_i \cap S$ in order to reduce the costs $b(x'', x')$ associated with choosing profitable activities x'' from $M_i \cap T$. If player k chooses some subset X_k of M_k for each $k \neq i$, then the problem for player i is to minimize

$$d_i(X) = \sum_{x \in X_i \cap S} c_x - \sum_{x \in X_i \cap T} r_x + \sum_{x' \in X_i} \sum_{x'' \in M \setminus X} b(x', x'')$$

over all subsets X_i of M_i where $X = \cup_{k=1}^n X_k$. This can be formulated as a minimum cut game as follows. Construct a network with a source s, a sink t, and m other nodes corresponding to each of the activities of M. For each player i define a nonnegative capacity function $c_i(x', x'')$ on each ordered pair of nodes (x', x'') from among $M \cup \{s\} \cup \{t\}$ so that $c_i(s, x) = r_x$ if x is in $M_i \cap T$, $c_i(x, t) = c_x$ if x is in $M_i \cap S$, $c_i(x', x'') = b(x', x'')$ if x' is in M_i and

x'' is in M, and $c_i(x', x'') = 0$ otherwise. For each player i, the associated cut capacity function for each subset X of M is

$$\begin{aligned} g_i(X) &= \textstyle\sum_{x' \in X_i \cup \{s\}} \sum_{x'' \in (M \setminus X) \cup \{t\}} c_i(x', x'') \\ &= \textstyle\sum_{x \in (M_i \setminus X_i) \cap T} r_x + \sum_{x \in X_i \cap S} c_x + \sum_{x' \in X_i} \sum_{x'' \in M \setminus X} b(x', x'') \\ &= d_i(X) + \textstyle\sum_{x \in M_i \cap T} r_x. \end{aligned}$$

The payoff function for firm i in this minimum cut game is

$$f_i(X) = -g_i(X) = -d_i(X) - \textstyle\sum_{x \in M_i \cap T} r_x.$$

This is a supermodular game because it is a minimum cut game, so a greatest (least) equilibrium point exists by Theorem 4.2.1. If the costs c_x and returns r_x are treated as parameters, then Theorem 4.2.2 implies that the greatest (least) equilibrium point increases with each r_x for x in $M \cap T$ and decreases with each c_x for x in $M \cap S$.

5

Cooperative Games

5.1 Introduction

This chapter considers the role of supermodularity and complementarity in cooperative games. The results primarily involve convex games, where the net return to any subset of players acting together (that is, the characteristic function evaluated for any coalition) is a supermodular function of the set of players. This introductory section proceeds first by briefly summarizing the content of the subsequent sections in the chapter and then by providing basic definitions and notation for the remaining sections.

Section 5.2 presents general properties of convex games. Much of the analysis is constructive and involves the remarkably simple, effective, and efficient greedy algorithm. Monotone comparative statics results are established, with the characteristic function and the set of players treated as parameters. Addressing issues of computational complexity and bounded rationality, combinatorial features of the polyhedral structure of the core are examined.

Section 5.3 gives examples of convex games. These involve a monopoly firm, monotonic surplus sharing, an aircraft landing fee game, and a trading game.

Section 5.4 presents and analyzes a general class of cooperative games, activity optimization games with complementarity, where each coalition maximizes a common return function over the levels of private activities of the members of the coalition and over the levels of public activities available to any coalition. These games offer a more refined model than the general games considered in previous sections because here optimal decision making by each coalition is incorporated in the model. Every game in this class is a convex game. Conditions are given for optimal activity levels, the extreme points of the core, and the Shapley value to increase with a parameter.

Section 5.5 gives examples of activity optimization games with complementarity. These are a general welfare game, a game in which firms jointly procure inputs for their production processes, a game with investment and production decisions, an activity selection game, and a waste disposal game. The examples of this section are all convex games as are the examples of Section 5.3, but these examples further model the making of optimal decisions by each possible coalition.

Section 5.6 presents two cooperative game models having an optimal decision problem for each coalition and with a complementarity property, but where the games are not generally convex games and the core may be empty. One is a network design game for a network that must handle multicommodity shipments by multiple users. The other is a max-min cooperative game model with the characteristic function for each coalition being the value of a certain noncooperative two-player zero-sum game. Properties of these examples contrast with properties of an activity optimization game with complementarity developed in Section 5.4, where the formal conditions of the model also describe an optimal decision problem for each coalition and include complementarity conditions but with these particular conditions implying that the game is a convex game.

The remainder of this section gives definitions and notation.

A **cooperative game with side payments** (abbreviated **cooperative game** in this chapter) is a pair **(N, f)** consisting of a finite set of **players** $N = \{1, \dots, n\}$ and a real-valued **characteristic function** $f(S)$ defined on every subset S of N. A **coalition** is an arbitrary subset of players who may operate together as a single organization, independent of the players not in the coalition. The characteristic function $f(S)$ is the net return that a coalition S could earn. It is assumed that $f(\emptyset) = 0$, so a game without players generates no net profit or cost. The characteristic function $f(S)$ is **superadditive** if

$$f(S') + f(S'') \le f(S' \cup S'')$$

for any two disjoint subsets S' and S'' of N. It is assumed that the characteristic function is superadditive, so any two disjoint groups of players can achieve at least as great a net profit by operating together as by operating separately. (In subsequent discussions of properties for a characteristic function in this chapter, recall the equivalence in part (d) of Example 2.2.1 of a function on $\mathcal{P}(N)$ and a function of the indicator vector in R^n, where $n = |N|$, of sets in $\mathcal{P}(N)$.) Given a cooperative game (N, f), for any subset N' of the players N the cooperative game (N', f) involving only the players N' and having the same characteristic function $f(S)$ on subsets of N' is a **subgame**.

If (N, f) is a cooperative game and the characteristic function $f(S)$ is supermodular on the subsets $\mathcal{P}(N)$ of N, then (N, f) is a **convex game**. (The term "convex" is commonly used in the game theory literature to mean supermodular, and hence the term "convex game." This monograph follows the literature in using the term "convex game" because it offers no ambiguity, but to avoid confusion the term "convex" is used herein only with its usual and rather different meaning for real-valued functions on R^n.) By Theorem 2.6.1 and Corollary 2.6.1, (N, f) is a convex game if and only if the marginal value of any player i to any coalition S not including i (that is, $f(S \cup \{i\}) - f(S)$) increases with S, so the incentive for a player to join a coalition increases with the membership of the coalition. If the characteristic function $f(S)$ is supermodular and $f(\emptyset) = 0$, then $f(S)$ is superadditive.

In a cooperative game (N, f), each player i may receive a **payoff** y_i. A vector $y = (y_1, \ldots, y_n)$ composed of a prospective payoff y_i for each player i in N is a **payoff vector**. A payoff vector y is **acceptable** if $\sum_{i \in S} y_i \geq f(S)$ for each subset S of the players N; that is, if the payoff vector provides a total payoff to each subset of players that is at least as great as the net return that the members of that subset could earn by forming a coalition and operating independently of the other players. A payoff vector y is **feasible** if $\sum_{i \in N} y_i = f(N)$; that is, if the total payoff to all players equals what the coalition of all players N could earn. The **core** of a cooperative game (N, f) is the set

$$\{y : y \in R^n, \textstyle\sum_{i \in N} y_i = f(N), \text{ and } \sum_{i \in S} y_i \geq f(S) \text{ for each } S \subseteq N\}$$

of all payoff vectors that are both acceptable and feasible. Given a payoff vector in the core, the coalition of all players N could form and distribute its entire earnings as payoffs to its members in such a way that no subset of the players would find it preferable to break off from the coalition N and form its own coalition. Sharkey [1982c] emphasizes the role of the core in the study of natural monopoly and related issues. The core of an arbitrary cooperative game need not be nonempty. The core of a cooperative game (N, f) is **large** if for each acceptable payoff vector y'' there exists a payoff vector y' in the core with $y' \leq y''$. If one wants a payoff vector in the core and if there is a given vector y'' such that the payoff to each player i cannot exceed y''_i, then the desired payoff vector does not exist if y'' is not acceptable and the desired payoff vector exists for any acceptable y'' if and only if the core is large. The cooperative game (N, f) has a **totally large core** if each subgame (that is, (N', f) for each subset N' of N) has a large core.

It is sometimes natural to consider a **cooperative cost game**, which is a pair (N, c) consisting of a finite set of players $N = \{1, \ldots, n\}$ and a real-valued cost function $c(S)$ defined on every subset S of N. The cost function $c(S)$ is the net cost that a coalition S would have. It is assumed that $c(\emptyset) = 0$. A cooperative cost game (N, c) can be transformed into an equivalent cooperative game $(N, -c)$, so only cooperative games and not cooperative cost games are considered explicitly in this chapter. See Subsection 5.3.1 and Subsection 5.3.3 for examples of cooperative cost games.

Given a set $M = \{1, \ldots, m\}$, a **polymatroid rank function** is a function $g(S)$ on the collection $\mathcal{P}(M)$ of all subsets of M such that $g(\emptyset) = 0$ and $g(S)$ is increasing and submodular on $\mathcal{P}(M)$. For a set $M = \{1, \ldots, m\}$ and a polymatroid rank function $g(S)$ on the subsets $\mathcal{P}(M)$ of M, a **polymatroid** is a set of the form

$$\{y : y \in R^m, y \geq 0, \textstyle\sum_{i \in S} y_i \leq g(S) \text{ for each } S \subseteq M\} \tag{5.1.1}$$

and the **base** of the polymatroid is the set

$$\{y : y \in R^m, y \geq 0, \textstyle\sum_{i \in S} y_i \leq g(S) \text{ for each } S \subseteq M, \sum_{i \in M} y_i = g(M)\}.$$

There is a close relationship between (the bases of) polymatroids and the cores of convex games, as Shapley [1971] notes.

Consider any cooperative game (N, f). A **permutation** π of the set of players $N = \{1, \ldots, n\}$ is a function $\pi(j)$ from $\{1, \ldots, n\}$ onto N. Let Π be the collection of all $n!$ permutations of N. For a permutation π of the players N, let $S(\pi, j) = \{\pi(1), \ldots, \pi(j)\}$ for $j = 0, 1, \ldots, n$. For a permutation π of the players N, the **greedy algorithm** generates the payoff vector $y(\pi)$ defined by $y(\pi)_{\pi(j)} = f(S(\pi, j)) - f(S(\pi, j-1))$ for $j = 1, \ldots, n$. For a given permutation π, the greedy algorithm gives to player $\pi(j)$ a payoff equal to the marginal value of that player when added to a coalition consisting of the players $S(\pi, j-1)$. Shapley [1971] introduces this greedy algorithm for the core of a convex game, and Edmonds [1970] introduces a similar greedy algorithm for a polymatroid.

For a cooperative game (N, f), Shapley [1953] shows that there exists a unique payoff vector, the **Shapley value**, satisfying the axiomatic properties of symmetry (that is, the payoff vector is not affected by permutations of the players), efficiency (that is, the sum of the payoffs to all players is $f(N)$ and the payoff is 0 to each player i with $f(S) = f(S \cup \{i\})$ for every subset S of $N \setminus \{i\}$), and aggregation (that is, the sum of the payoff vectors for cooperative games (N, f_1) and (N, f_2) equals the payoff vector for the cooperative game $(N, f_1 + f_2)$). The Shapley value assigns a payoff

$$\textstyle\sum_{S \subseteq N \setminus \{i\}} (|S|!(n - |S| - 1)!/n!)(f(S \cup \{i\}) - f(S))$$

to each player i. The Shapley value gives a payoff to player i that is the average over all coalitions S not containing i of the marginal value $(f(S \cup \{i\}) - f(S))$ of player i when added to coalition S. Equivalently, the Shapley value equals $(1/n!) \sum_{\pi \in \Pi} y(\pi)$, which assigns to each player i the average received by player i from all $n!$ payoff vectors $y(\pi)$. For an arbitrary cooperative game, the Shapley value is feasible but need not be acceptable and hence need not be in the core.

This chapter emphasizes the core and the Shapley value because of the importance of these solution concepts and because of the clear role of supermodularity and complementarity in their analysis. For a broader view of cooperative games, see Ichiishi [1993], Moulin [1988], and Shubik [1981].

5.2 Convex Games

This section studies general properties of convex games. Subsection 5.2.1 examines the core of a convex game, with the greedy algorithm and its extremal properties being key to the analysis. Subsection 5.2.2 develops parametric properties of cooperative games with respect to changes in the characteristic function and the set of players. Subsection 5.2.3 considers the polyhedral structure of the core of a convex game, with a focus on issues of computational complexity and bounded rationality.

5.2.1 The Core and the Greedy Algorithm

Theorem 5.2.1, from Shapley [1971], shows that for a convex game each payoff vector generated by the greedy algorithm is in the core and consequently the Shapley value is also in the core.

Theorem 5.2.1. *Suppose that (N, f) is a convex game.*
(a) *For each permutation π of the set of players N, the payoff vector $y(\pi)$ generated by the greedy algorithm is in the core.*
(b) *The core is nonempty.*
(c) *The Shapley value is in the core.*

Proof. Pick any permutation π of N, and let $y(\pi)$ be the corresponding payoff vector generated by the greedy algorithm. Then

$$\begin{aligned}\textstyle\sum_{i\in N} y(\pi)_i &= \textstyle\sum_{j=1}^{n} y(\pi)_{\pi(j)} \\ &= \textstyle\sum_{j=1}^{n}(f(S(\pi, j)) - f(S(\pi, j-1))) \\ &= f(S(\pi, n)) - f(S(\pi, 0)) \\ &= f(N) - f(\emptyset) \\ &= f(N).\end{aligned}$$

Let S' be any nonempty subset of the players N, let j'' be the largest j with $\pi(j)$ in S', and let j' be the smallest j with $\pi(j)$ in S'. With the inequality below following from the supermodularity of the characteristic function $f(S)$,

$$\begin{aligned}\textstyle\sum_{i\in S'} y(\pi)_i &= \textstyle\sum_{\{j:1\le j\le n,\pi(j)\in S'\}} y(\pi)_{\pi(j)}\\ &= \textstyle\sum_{\{j:1\le j\le n,\pi(j)\in S'\}}(f(S(\pi,j))-f(S(\pi,j-1)))\\ &\ge \textstyle\sum_{\{j:1\le j\le n,\pi(j)\in S'\}}(f(S'\cap S(\pi,j))-f(S'\cap S(\pi,j-1)))\\ &= f(S'\cap S(\pi,j''))-f(S'\cap S(\pi,j'-1))\\ &= f(S')-f(\emptyset)\\ &= f(S').\end{aligned}$$

Thus $y(\pi)$ is in the core, and the proof of part (a) is complete.

Part (b) is immediate from part (a).

Let $y_i' = \sum_{S\subseteq N\setminus\{i\}}(|S|!(n-|S|-1)!/n!)(f(S\cup\{i\})-f(S))$ be the payoff to player i according to the Shapley value, and $y'=(y_1',\ldots,y_n')$. Letting $y(\pi)$ be the payoff vector generated by the greedy algorithm with the permutation π of the players, observe that

$$y_i' = \textstyle\sum_{S\subseteq N\setminus\{i\}}(|S|!(n-|S|-1)!/n!)(f(S\cup\{i\})-f(S)) = \sum_{\pi\in\Pi}(1/n!)y(\pi)_i$$

for each i. For any subset S' of N,

$$\begin{aligned}\textstyle\sum_{i\in S'} y_i' &= \textstyle\sum_{i\in S'}\sum_{S\subseteq N\setminus\{i\}}(|S|!(n-|S|-1)!/n!)(f(S\cup\{i\})-f(S))\\ &= \textstyle\sum_{i\in S'}\sum_{\pi\in\Pi}(1/n!)y(\pi)_i\\ &= (1/n!)\textstyle\sum_{\pi\in\Pi}\sum_{i\in S'} y(\pi)_i\\ &\ge (1/n!)\textstyle\sum_{\pi\in\Pi} f(S')\\ &= f(S')\end{aligned}$$

with the inequality following because $y(\pi)$ is in the core by part (a). Also,

$$\begin{aligned}\textstyle\sum_{i\in N} y_i' &= \textstyle\sum_{i\in N}\sum_{S\subseteq N\setminus\{i\}}(|S|!(n-|S|-1)!/n!)(f(S\cup\{i\})-f(S))\\ &= \textstyle\sum_{i\in N}\sum_{\pi\in\Pi}(1/n!)y(\pi)_i\\ &= (1/n!)\textstyle\sum_{\pi\in\Pi}\sum_{i\in N} y(\pi)_i\\ &= (1/n!)\textstyle\sum_{\pi\in\Pi} f(N)\\ &= f(N)\end{aligned}$$

with the penultimate equality following because $y(\pi)$ is in the core by part (a). Therefore, the Shapley value is in the core and part (c) holds. □

Lemma 5.2.1 shows that each payoff vector generated by the greedy algorithm has certain extremal properties. For a payoff vector generated by the greedy algorithm with a permutation π, the n players are ordered so that this particular payoff vector favors their respective interests in the order $\pi(n), \pi(n-1), \dots, \pi(1)$. Theorem 5.2.2 further develops extremal properties of payoff vectors generated by the greedy algorithm. Part (a) of Lemma 5.2.1 is noted in Shapley [1971].

Lemma 5.2.1. *Suppose that (N, f) is a convex game, π is any permutation of the players N, and $y(\pi)$ is the payoff vector generated by the greedy algorithm with the permutation π. Then*
(a) $\sum_{i\in S(\pi,j)} y(\pi)_i = f(S(\pi, j))$ *for* $j = 1, \dots, n$;
(b) $y(\pi)$ *simultaneously maximizes* $\sum_{i\in N\setminus S(\pi,j)} y_i$ *for* $j = 1, \dots, n$ *over all payoff vectors y in the core; and*
(c) $y(\pi)$ *lexicographically maximizes* $(y_{\pi(n)}, y_{\pi(n-1)}, \dots, y_{\pi(1)})$ *over all payoff vectors y in the core.*

Proof. Pick any integer j with $1 \le j \le n$. Then

$$\begin{aligned}\textstyle\sum_{i\in S(\pi,j)} y(\pi)_i &= \textstyle\sum_{k=1}^{j} y(\pi)_{\pi(k)}\\ &= \textstyle\sum_{k=1}^{j}(f(S(\pi, k)) - f(S(\pi, k-1)))\\ &= f(S(\pi, j)) - f(S(\pi, 0))\\ &= f(S(\pi, j)) - f(\emptyset)\\ &= f(S(\pi, j)),\end{aligned}$$

establishing part (a).

Let y' be any vector in the core, and pick any integer j with $1 \le j \le n$. Then

$$\begin{aligned}\textstyle\sum_{i\in N\setminus S(\pi,j)} y(\pi)_i &= \textstyle\sum_{i\in N} y(\pi)_i - \sum_{i\in S(\pi,j)} y(\pi)_i\\ &= f(N) - f(S(\pi, j))\\ &\ge \textstyle\sum_{i\in N} y'_i - \sum_{i\in S(\pi,j)} y'_i\\ &= \textstyle\sum_{i\in N\setminus S(\pi,j)} y'_i,\end{aligned}$$

where the second equality follows from part (a) of Theorem 5.2.1 and from part (a) and the inequality follows from part (a) of Theorem 5.2.1 and because y' is in the core. Thus, part (b) holds.

Pick any payoff vector y' in the core with $y' \ne y(\pi)$. Let the integer j' be such that $y'_{\pi(j)} = y(\pi)_{\pi(j)}$ for $j = j' + 1, \dots, n$ and $y'_{\pi(j')} \ne y(\pi)_{\pi(j')}$. To complete the proof of part (c), it suffices to show that $y(\pi)_{\pi(j')} > y'_{\pi(j')}$ This holds because $y'_{\pi(j')} \neq y(\pi)_{\pi(j')}$ and

$$\begin{aligned}
y(\pi)_{\pi(j')} - y'_{\pi(j')} &= \textstyle\sum_{j=j'}^{n} y(\pi)_{\pi(j)} - \sum_{j=j'}^{n} y'_{\pi(j)} \\
&= \textstyle\sum_{i\in N\setminus S(\pi,j'-1)} y(\pi)_i - \sum_{i\in N\setminus S(\pi,j'-1)} y'_i \\
&= \textstyle\sum_{i\in N} y(\pi)_i - \sum_{i\in S(\pi,j'-1)} y(\pi)_i - \sum_{i\in N} y'_i + \sum_{i\in S(\pi,j'-1)} y'_i \\
&= \textstyle\sum_{i\in S(\pi,j'-1)} y'_i - \sum_{i\in S(\pi,j'-1)} y(\pi)_i \\
&= \textstyle\sum_{i\in S(\pi,j'-1)} y'_i - f(S(\pi, j'-1)) \\
&\geq 0,
\end{aligned}$$

where the fourth equality follows because $y(\pi)$ and y' are in the core (using part (a) of Theorem 5.2.1), the fifth equality follows from part (a), and the inequality follows because y' is in the core. □

By Lemma 5.2.2, from Shapley [1971], strict supermodularity of the characteristic function is a necessary and sufficient condition for the greedy algorithm to generate distinct payoff vectors with the $n!$ distinct permutations of the players in a convex game.

Lemma 5.2.2. *For a convex game (N, f), the characteristic function $f(S)$ is strictly supermodular on $\mathcal{P}(N)$ if and only if the $n!$ payoff vectors generated by the greedy algorithm with the $n!$ different permutations of the players N are distinct.*

Proof. Suppose that $f(S)$ is strictly supermodular. Pick any two distinct permutations π' and π'' of N. Let the positive integer j' be such that $\pi'(j') \neq \pi''(j')$ and $\pi'(j) = \pi''(j)$ for $j = 1, \ldots, j'-1$. Let the positive integer j'' be such that $\pi'(j') = \pi''(j'')$, so $j' < j''$ and $S(\pi', j'-1) = S(\pi'', j'-1) \subset S(\pi'', j''-1)$. With the strict inequality following from the strict supermodularity of $f(S)$,

$$\begin{aligned}
y(\pi')_{\pi'(j')} &= f(S(\pi', j')) - f(S(\pi', j'-1)) \\
&< f(S(\pi'', j'')) - f(S(\pi'', j''-1)) \\
&= y(\pi'')_{\pi''(j'')} \\
&= y(\pi'')_{\pi'(j')}
\end{aligned}$$

and so $y(\pi') \neq y(\pi'')$.

Now suppose that the payoff vectors generated by the greedy algorithm with the different permutations of N are distinct. Pick any subset S' of N with $|S'| \leq n-2$, and pick distinct i' and i'' in $N \setminus S'$. Let π' be any permutation of N with $\{\pi'(1), \ldots, \pi'(|S'|)\} = S'$, $\pi'(|S'|+1) = i'$, $\pi'(|S'|+2) = i''$, and $\{\pi'(|S'|+3), \ldots, \pi'(|N|)\} = N \setminus (S' \cup \{i', i''\})$. Let π'' be the permutation of

N with $\pi''(|S'|+1) = i''$, $\pi''(|S'|+2) = i'$, and $\pi''(j) = \pi'(j)$ for each j with $j \neq |S'|+1$ and $j \neq |S'|+2$. Because π' and π'' are distinct, $y(\pi') \neq y(\pi'')$. Therefore,

$$\begin{aligned}
&(f(S' \cup \{i', i''\}) - f(S' \cup \{i''\})) - (f(S' \cup \{i'\}) - f(S')) \\
&\quad = (f(S(\pi'', |S'|+2)) - f(S(\pi'', |S'|+1))) \\
&\qquad - (f(S(\pi', |S'|+1)) - f(S(\pi', |S'|))) \\
&\quad = y(\pi'')_{\pi''(|S'|+2)} - y(\pi')_{\pi'(|S'|+1)} \\
&\quad = y(\pi'')_{i'} - y(\pi')_{i'} \\
&\quad \neq 0.
\end{aligned} \tag{5.2.1}$$

Because (N, f) is a convex game, $f(S)$ is supermodular and so $f(S)$ has increasing differences by Theorem 2.6.1. Then $f(S)$ has strictly increasing differences by (5.2.1), and so $f(S)$ is strictly supermodular by Corollary 2.6.1. □

Suppose that for a cooperative game (N, f) there is a hierarchy of m groups of players ordered in terms of their power to choose a payoff vector y from the core. The k^{th} most powerful group has a linear objective function $r^{(k)} \cdot y$ for y in the core, where $r^{(k)}$ is an n-vector for $k = 1, \ldots, m$. Because of the hierarchy, the most powerful group requires that any payoff vector chosen from the core maximizes $r^{(1)} \cdot y$ over the core; the second most powerful group then requires that any payoff vector chosen maximizes $r^{(2)} \cdot y$ over those vectors in the core that maximize $r^{(1)} \cdot y$ over the core; and so on. The problem of the m groups is to lexicographically maximize $(r^{(1)} \cdot y, \ldots, r^{(m)} \cdot y)$ over payoff vectors y in the core; that is, to determine a payoff vector y' in the core such that $(r^{(1)} \cdot y, \ldots, r^{(m)} \cdot y) \leq_{lex} (r^{(1)} \cdot y', \ldots, r^{(m)} \cdot y')$ for each payoff vector y in the core. If $m = 1$, this reduces to maximizing a linear function over the core. Edmonds [1970] shows that one can maximize an arbitrary linear function over a polymatroid by implementing the greedy algorithm with a suitable permutation. Similarly, one can maximize an arbitrary linear function over the core of a convex game by implementing the greedy algorithm with a suitable permutation. Indeed, Theorem 5.2.2, from Topkis [1987], shows that for a convex game the greedy algorithm implemented with a suitable permutation of the players generates the lexicographic maximum of an m-vector of linear functions over the core. One can demonstrate a similar result for polymatroids. The solution given in Theorem 5.2.2 for lexicographic maximization over the core of a convex game underscores the greedy algorithm's property of choosing a payoff vector that favors certain groups of players. For $i = 1, \ldots, n$, let

$r^i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(m)})$. In implementing Theorem 5.2.2, observe that, using properties of the lexicographic ordering relation, it is straightforward to find a permutation π of the players with $r^{\pi(1)} \leq_{lex} r^{\pi(2)} \leq_{lex} \cdots \leq_{lex} r^{\pi(n)}$.

Theorem 5.2.2. *If (N, f) is a convex game, $r^{(k)} = (r_1^{(k)}, \ldots, r_n^{(k)})$ is in R^n for $k = 1, \ldots, m$, $r^i = (r_i^{(1)}, \ldots, r_i^{(m)})$ for $i = 1, \ldots, n$, and π is a permutation of the players such that $r^{\pi(1)} \leq_{lex} r^{\pi(2)} \leq_{lex} \cdots \leq_{lex} r^{\pi(n)}$, then the payoff vector $y(\pi)$ lexicographically maximizes $(r^{(1)} \cdot y, \ldots, r^{(m)} \cdot y)$ over all payoff vectors y in the core.*

Proof. Pick any payoff vector y' in the core. For $j = 2, \ldots, n$, $\sum_{h=1}^{j-1} y'_{\pi(h)} \geq f(S(\pi, j-1)) = \sum_{h=1}^{j-1} y(\pi)_{\pi(h)}$ because y' is in the core and by part (a) of Lemma 5.2.1. Because y' is in the core by assumption and by part (a) of Theorem 5.2.1, $\sum_{j=1}^{n} y'_{\pi(j)} = f(N) = \sum_{j=1}^{n} y(\pi)_{\pi(j)}$ and so $\sum_{h=j}^{n} y'_{\pi(h)} \leq \sum_{h=j}^{n} y(\pi)_{\pi(h)}$ for $j = 2, \ldots, n$. Therefore,

$$
\begin{aligned}
&(r^{(1)} \cdot y', \ldots, r^{(m)} \cdot y') \\
&= \textstyle\sum_{j=1}^{n} r^{\pi(j)} y'_{\pi(j)} \\
&= r^{\pi(1)}\big(\textstyle\sum_{j=1}^{n} y'_{\pi(j)}\big) + \textstyle\sum_{j=2}^{n}(r^{\pi(j)} - r^{\pi(j-1)})\big(\textstyle\sum_{h=j}^{n} y'_{\pi(h)}\big) \\
&\leq_{lex} r^{\pi(1)}\big(\textstyle\sum_{j=1}^{n} y(\pi)_{\pi(j)}\big) + \textstyle\sum_{j=2}^{n}(r^{\pi(j)} - r^{\pi(j-1)})\big(\textstyle\sum_{h=j}^{n} y(\pi)_{\pi(h)}\big) \\
&= \textstyle\sum_{j=1}^{n} r^{\pi(j)} y(\pi)_{\pi(j)} \\
&= (r^{(1)} \cdot y(\pi), \ldots, r^{(m)} \cdot y(\pi)). \quad \square
\end{aligned}
$$

Theorem 5.2.3, from Ichiishi [1981], shows as a converse to part (a) of Theorem 5.2.1 that if every vector generated by the greedy algorithm for some cooperative game is in the core then that game must be a convex game. Similarly, Dunstan and Welsh [1973] show that, for a set of the form (5.1.1), $g(S)$ must be submodular if the greedy algorithm with a suitable permutation generates the maximum of an arbitrary linear function over that set.

Theorem 5.2.3. *If (N, f) is a cooperative game and the payoff vector generated by the greedy algorithm for each permutation of the players is in the core, then the cooperative game (N, f) is a convex game.*

Proof. Let S' and S'' be any subsets of N. Construct the permutation π of N so that $\{\pi(1), \ldots, \pi(|S' \cap S''|)\} = S' \cap S''$, $\{\pi(|S' \cap S''|+1), \ldots, \pi(|S'|)\} = S' \setminus S''$, $\{\pi(|S'|+1), \ldots, \pi(|S' \cup S''|)\} = S'' \setminus S'$, and $\{\pi(|S' \cup S''|+1), \ldots, \pi(n)\} = N \setminus (S' \cup S'')$. Let $y(\pi)$ be the payoff vector generated by the greedy algorithm with the permutation π. Because the payoff vector $y(\pi)$ is in the core by hypothesis,

$$\begin{aligned}
f(S'') &\le \textstyle\sum_{i \in S''} y(\pi)_i \\
&= \textstyle\sum_{i \in S' \cap S''} y(\pi)_i + \sum_{i \in S'' \setminus S'} y(\pi)_i \\
&= \textstyle\sum_{j=1}^{|S' \cap S''|} (f(S(\pi, j)) - f(S(\pi, j-1))) \\
&\quad + \textstyle\sum_{j=|S'|+1}^{|S' \cup S''|} (f(S(\pi, j)) - f(S(\pi, j-1))) \\
&= f(S(\pi, |S' \cap S''|)) - f(S(\pi, 0)) + f(S(\pi, |S' \cup S''|)) - f(S(\pi, |S'|)) \\
&= f(S' \cap S'') - f(\emptyset) + f(S' \cup S'') - f(S') \\
&= f(S' \cap S'') + f(S' \cup S'') - f(S').
\end{aligned}$$

Thus $f(S)$ is supermodular and (N, f) is a convex game. □

Part (a) of Theorem 5.2.4, from Shapley [1971], shows that for a convex game the set of payoff vectors generated by the greedy algorithm using all permutations of the players is the set of all extreme points of the core. Edmonds [1970] establishes a similar result for polymatroids. Therefore, a convex game with n players has no more than $n!$ extreme points and, by Lemma 5.2.2, has exactly $n!$ extreme points if and only if the characteristic function is strictly supermodular. A **convex combination** of a nonempty finite collection of vectors in R^n is a linear combination of those vectors where the coefficients in the linear combination are nonnegative and sum to 1.

Theorem 5.2.4. *Consider a convex game.*
(a) *A payoff vector is an extreme point of the core if and only if it is generated by the greedy algorithm with some permutation of the players.*
(b) *A payoff vector is in the core if and only if it is a convex combination of (at most $n+1$) payoff vectors generated by the greedy algorithm with some permutation of the players.*

Proof. Let (N, f) be a convex game.

Pick any permutation π of the players. Suppose that $y(\pi)$ is not an extreme point of the core, so there exist distinct y' and y'' in the core and α with $0 < \alpha < 1$ such that $y(\pi) = \alpha y' + (1-\alpha)y''$. Let the integer j' be such that $y'_{\pi(j')} \ne y''_{\pi(j')}$ and $y'_{\pi(j)} = y''_{\pi(j)}$ for $j = 1, \ldots, j'-1$. Without loss of generality, say $y'_{\pi(j')} < y''_{\pi(j')}$. Thus $y'_{\pi(j')} < y(\pi)_{\pi(j')} < y''_{\pi(j')}$ and $y'_{\pi(j)} = y(\pi)_{\pi(j)} = y''_{\pi(j)}$ for $j = 1, \ldots, j'-1$. By part (a) of Lemma 5.2.1, $f(S(\pi, j')) = \sum_{i \in S(\pi, j')} y(\pi)_i > \sum_{i \in S(\pi, j')} y'_i$, which contradicts y' being in the core. Therefore, $y(\pi)$ is an extreme point of the core.

Now pick any extreme point y' of the core. Suppose that y' is not generated by the greedy algorithm for any permutation of N. Because $y(\pi)$ is in the core for each permutation π of N by part (a) of Theorem 5.2.1 and y' is an extreme point distinct from all $y(\pi)$, y' is not a convex combination of $\{y(\pi) : \pi \in \Pi\}$. Therefore, there exists a vector r in R^n with $r \cdot y(\pi) < r \cdot y'$ for each permutation π of N (Rockafellar [1970]). But this contradicts the result of Theorem 5.2.2 that some vector generated by the greedy algorithm maximizes $r \cdot y$ over all y in the core. Therefore, y' is generated by the greedy algorithm for some permutation of N, concluding the proof of part (a).

Any vector in a compact convex subset of R^n is a convex combination of at most $n + 1$ extreme points of the set (Rockafellar [1970]). Part (b) follows from this property together with part (a). □

For a cooperative game (N, f), a function $g(S)$ defined on all subsets S of N with $g(\emptyset) = g(N) = 0$, and a real ϵ, define $f_{g,\epsilon}(S) = f(S) + \epsilon g(S)$ for each subset S of N. The core of the cooperative game $(N, f_{g,\epsilon})$ is the ***g*-ϵ-core** of (N, f). Natural examples for $g(S)$ include $g(S) = 1$ and $g(S) = |S|$ for each subset S of N with $1 \le |S| \le n - 1$. If $\epsilon \ge 0$ and $g(S) \ge 0$ for each subset S of N, then a vector in the g-ϵ-core gives to each coalition S a payoff that exceeds its lower bound from core payoffs, $f(S)$, by at least $\epsilon g(S)$. (In a convex game, $f(S)$ is the greatest lower bound on the payoff to a coalition S from core payoffs by part (a) of Theorem 5.2.1 and part (a) of Lemma 5.2.1.) In order to mitigate the extremal properties of payoff vectors generated by the greedy algorithm as expressed in Lemma 5.2.1 and Theorem 5.2.2, it might sometimes be preferable to seek a vector in the g-ϵ-core of a convex game rather than use the greedy algorithm to generate an extreme point of the core. For a convex game (N, f) and a given $g(S)$, Theorem 5.2.5, from Topkis [1987], shows how to compute the largest ϵ for which $(N, f_{g,\epsilon})$ is a convex game. For ϵ no greater than that number, the greedy algorithm applied to the convex game $(N, f_{g,\epsilon})$ using all permutations π of the players N generates all extreme points of the g-ϵ-core of (N, f) by part (a) of Theorem 5.2.4. If ϵ is strictly greater than the bound given in Theorem 5.2.5, then $(N, f_{g,\epsilon})$ is not a convex game and so Theorem 5.2.3 implies that the g-ϵ-core of (N, f) does not contain every vector generated by applying the greedy algorithm to $(N, f_{g,\epsilon})$. In the statement of Theorem 5.2.5, observe the convention that the minimum of an empty collection of real numbers is taken to be $+\infty$.

Theorem 5.2.5. *Suppose that (N, f) is a convex game, $g(S)$ is supermodular on every collection of subsets of N that is closed under union and intersection and that includes neither $\emptyset$ nor N, $g(\emptyset) = g(N) = 0$, and $\epsilon \ge 0$. Let ϵ_1 be the minimum of* $-(f(\{i', i''\}) + f(\emptyset) - f(\{i'\}) - f(\{i''\}))/(g(\{i', i''\}) + g(\emptyset) -$

$g(\{i'\}) - g(\{i''\}))$ over all distinct players i' and i'' in N with $g(\{i', i''\}) + g(\emptyset) - g(\{i'\}) - g(\{i''\}) < 0$, and let ϵ_2 be the minimum of $-(f(N) + f(N \setminus \{i', i''\}) - f(N \setminus \{i''\}) - f(N \setminus \{i'\}))/(g(N) + g(N \setminus \{i', i''\}) - g(N \setminus \{i''\}) - g(N \setminus \{i'\}))$ over all distinct players i' and i'' in N with $g(N) + g(N \setminus \{i', i''\}) - g(N \setminus \{i''\}) - g(N \setminus \{i'\}) < 0$. Then $f_{g,\epsilon}(S)$ is supermodular if and only if $\epsilon \leq \min\{\epsilon_1, \epsilon_2\}$.

Proof. Pick any distinct i' and i'' in N and any subset S' of $N \setminus \{i', i''\}$. By Theorem 2.6.1 and Corollary 2.6.1, $f_{g,\epsilon}(S)$ is supermodular if and only if

$$f_{g,\epsilon}(S' \cup \{i', i''\}) + f_{g,\epsilon}(S') - f_{g,\epsilon}(S' \cup \{i'\}) - f_{g,\epsilon}(S' \cup \{i''\}) \geq 0.$$

If $1 \leq |S'| \leq n - 3$, then

$$\begin{aligned} &f_{g,\epsilon}(S' \cup \{i', i''\}) + f_{g,\epsilon}(S') - f_{g,\epsilon}(S' \cup \{i'\}) - f_{g,\epsilon}(S' \cup \{i''\}) \\ &\quad = f(S' \cup \{i', i''\}) + f(S') - f(S' \cup \{i'\}) - f(S' \cup \{i''\}) \\ &\qquad + \epsilon(g(S' \cup \{i', i''\}) + g(S') - g(S' \cup \{i'\}) - g(S' \cup \{i''\})) \\ &\quad \geq 0 \end{aligned}$$

by hypotheses. If $|S'| = 0$, then $S' = \emptyset$ and

$$\begin{aligned} &f_{g,\epsilon}(S' \cup \{i', i''\}) + f_{g,\epsilon}(S') - f_{g,\epsilon}(S' \cup \{i'\}) - f_{g,\epsilon}(S' \cup \{i''\}) \\ &\quad = f(\{i', i''\}) + f(\emptyset) - f(\{i'\}) - f(\{i''\}) \\ &\qquad + \epsilon(g(\{i', i''\}) + g(\emptyset) - g(\{i'\}) - g(\{i''\})) \\ &\quad \geq 0 \end{aligned}$$

for all distinct i' and i'' in N if and only if

$$\begin{aligned} \epsilon \leq \min\{&-(f(\{i', i''\}) + f(\emptyset) - f(\{i'\}) - f(\{i''\}))/ \\ &(g(\{i', i''\}) + g(\emptyset) - g(\{i'\}) - g(\{i''\})) : \\ &i' \in N, i'' \in N, i' \neq i'', g(\{i', i''\}) + g(\emptyset) - g(\{i'\}) - g(\{i''\}) < 0\}. \end{aligned}$$

If $|S'| = n - 2$ and $\{i', i''\} = N \setminus S'$, then

$$\begin{aligned} &f_{g,\epsilon}(S' \cup \{i', i''\}) + f_{g,\epsilon}(S') - f_{g,\epsilon}(S' \cup \{i'\}) - f_{g,\epsilon}(S' \cup \{i''\}) \\ &\quad = f(N) + f(N \setminus \{i', i''\}) - f(N \setminus \{i''\}) - f(N \setminus \{i'\}) \\ &\qquad + \epsilon(g(N) + g(N \setminus \{i', i''\}) - g(N \setminus \{i''\}) - g(N \setminus \{i'\})) \\ &\quad \geq 0 \end{aligned}$$

for all distinct i' and i'' in N if and only if

$$\begin{aligned}\epsilon \le \min\{&-(f(N) + f(N \setminus \{i', i''\}) - f(N \setminus \{i''\}) - f(N \setminus \{i'\}))/\\ &(g(N) + g(N \setminus \{i', i''\}) - g(N \setminus \{i''\}) - g(N \setminus \{i'\})) :\\ &i' \in N, i'' \in N, i' \neq i'', g(N) + g(N \setminus \{i', i''\})\\ &- g(N \setminus \{i''\}) - g(N \setminus \{i'\}) < 0\}. \quad \square\end{aligned}$$

Sharkey [1982a] shows that the core of a convex game is large. Moulin [1990] sharpens this implication and provides a converse by characterizing convex games as those cooperative games with a totally large core. Theorem 5.2.6 states this equivalence.

Theorem 5.2.6. *A cooperative game is a convex game if and only if it has a totally large core.*

Proof. Because each subgame of a convex game is a convex game, to establish necessity it suffices to show that a convex game has a large core. Suppose that (N, f) is a convex game. Let y'' be any acceptable payoff vector. For each subset S of N, define $g(S) = \max_{\{S': S \subseteq S' \subseteq N\}}(f(S') - \sum_{i \in S'} y''_i)$. Because $(f(S) - \sum_{i \in S} y''_i)$ is supermodular in S on $\mathcal{P}(N)$ by Theorem 2.6.4 and part (b) of Lemma 2.6.1, $g(S)$ is supermodular by Theorem 2.7.6. Because y'' is an acceptable payoff vector for the cooperative game (N, f), $g(S) \le 0$ for each subset S of N and $g(\emptyset) = 0$. Furthermore, $g(S)$ is decreasing in S. By part (b) of Theorem 5.2.1, the convex game (N, g) has a nonempty core. Pick any y' in the core of (N, g). For each i' in N, $\sum_{i \in N \setminus \{i'\}} y'_i \ge g(N \setminus \{i'\}) \ge g(N) = \sum_{i \in N} y'_i$ and so $y'_{i'} \le 0$. Let $y''' = y'' + y' \le y''$. For any subset S of N,

$$\begin{aligned}\textstyle\sum_{i \in S} y'''_i &= \textstyle\sum_{i \in S} y''_i + \sum_{i \in S} y'_i \ge \sum_{i \in S} y''_i + g(S)\\ &\ge \textstyle\sum_{i \in S} y''_i + f(S) - \sum_{i \in S} y''_i = f(S).\end{aligned}$$

Also,

$$\begin{aligned}\textstyle\sum_{i \in N} y'''_i &= \textstyle\sum_{i \in N} y''_i + \sum_{i \in N} y'_i = \sum_{i \in N} y''_i + g(N)\\ &= \textstyle\sum_{i \in N} y''_i + f(N) - \sum_{i \in N} y''_i = f(N).\end{aligned}$$

Therefore, y''' is in the core of (N, f) and so (N, f) has a large core.

Now suppose that (N, f) has a totally large core. Let S' and S'' be any subsets of N. Because (N, f) has a totally large core, there exists $\{y''_i : i \in S' \cap S''\}$ in the core of the subgame $(S' \cap S'', f)$. Define the elements of $\{y''_i : i \in (S' \cup S'') \setminus (S' \cap S'')\}$ to be large enough so that $\{y''_i : i \in S' \cup S''\}$ is acceptable for the subgame $(S' \cup S'', f)$. Because (N, f) has a totally large core, there exists $\{y'_i : i \in S' \cup S''\}$ in the core of the subgame $(S' \cup S'', f)$ with $y'_i \leq y''_i$ for each i in $S' \cup S''$. Then,

$$\begin{aligned} f(S') + f(S'') &\leq \textstyle\sum_{i\in S'} y'_i + \sum_{i\in S''} y'_i \\ &= \textstyle\sum_{i\in S'\cup S''} y'_i + \sum_{i\in S'\cap S''} y'_i \\ &\leq \textstyle\sum_{i\in S'\cup S''} y'_i + \sum_{i\in S'\cap S''} y''_i \\ &= f(S' \cup S'') + f(S' \cap S''). \end{aligned}$$

Therefore, (N, f) is a convex game. □

5.2.2 Games with a Parameter

Suppose that T is a partially ordered set and (N, f^t) is a cooperative game for each t in T, where the characteristic function $f^t(S)$ depends on the parameter t. This collection of cooperative games (N, f^t) for t in T has a **complementary parameter** t if $f^t(S)$ has increasing differences in (S, t) on $\mathcal{P}(N) \times T$; that is, if $f^t(S'') - f^t(S')$ is increasing in t on T for all subsets S' and S'' of N with S' being a subset of S''. The vector generated by the greedy algorithm given a permutation π of the players N and a parameter t in T is denoted $y^t(\pi)$. With a complementary parameter, the marginal value $f^t(S \cup \{i\}) - f^t(S)$ from adding any given player i to any given coalition S is increasing with the parameter t.

Theorem 5.2.7, from Topkis [1987], shows that the payoff vector generated by the greedy algorithm given any permutation of the players for a collection of cooperative games with a complementary parameter is increasing with the parameter. As a consequence, the Shapley value for each of a collection of cooperative games with a complementary parameter increases with the parameter. Another consequence is that any convex combination (with given coefficients) of the extreme points of a collection of convex games with a complementary parameter increases with the parameter.

Theorem 5.2.7. *Suppose that T is a partially ordered set, (N, f^t) is a cooperative game for each t in T, and t is a complementary parameter.*
(a) *The payoff vector $y^t(\pi)$ generated by the greedy algorithm is an increasing function of t on T for each permutation π of the players N.*
(b) *The Shapley value is an increasing function of t on T.*
(c) *If, in addition, (N, f^t) is a convex game for each t in T, then $\sum_{\pi\in\Pi} \alpha(\pi) y^t(\pi)$ is a linear function from $\{\alpha = (\alpha(\pi) : \pi \in \Pi) : \alpha(\pi) \geq 0$ for each $\pi \in \Pi$, $\sum_{\pi\in\Pi} \alpha(\pi) = 1\}$ onto the core of the convex game (N, f^t) for each t in T and $\sum_{\pi\in\Pi} \alpha(\pi) y^t(\pi)$ is an increasing function of t on T for each $\alpha \geq 0$.*

Proof. Pick any permutation π of N and any t' and t'' in T with $t' \preceq t''$. For $j = 1, \ldots, n$,

$$\begin{aligned} y^{t'}(\pi)_{\pi(j)} &= f^{t'}(S(\pi, j)) - f^{t'}(S(\pi, j-1)) \\ &\le f^{t''}(S(\pi, j)) - f^{t''}(S(\pi, j-1)) = y^{t''}(\pi)_{\pi(j)} \end{aligned}$$

because $f^t(S)$ has increasing differences in (S, t). Hence, part (a) holds.

Because the collection of cooperative games (N, f^t) has a complementarity parameter t for t in T, $f^t(S \cup \{i\}) - f^t(S)$ is increasing in t for each i in N and each subset S of $N \setminus \{i\}$. Each component of the Shapley value, $\sum_{S \subseteq N \setminus \{i\}}(|S|!(n - |S| - 1)!/n!)(f^t(S \cup \{i\}) - f^t(S))$, is therefore increasing in t and part (b) holds.

The first statement of part (c) follows from part (b) of Theorem 5.2.4. Part (a) then implies the second statement of part (c). □

Consider any parameterized collection of convex games (N, f^t) with the parameter t in a partially ordered set T and with t being a complementary parameter. Pick any parameter t' in T and any payoff vector $y^{t'}$ in the core of the convex game $(N, f^{t'})$. By part (b) of Theorem 5.2.4, there exist $\alpha(\pi) \ge 0$ for each permutation π of the players N such that $\sum_{\pi \in \Pi} \alpha(\pi) = 1$ and $y^{t'} = \sum_{\pi \in \Pi} \alpha(\pi) y^{t'}(\pi)$. For each t in T, let $y^t = \sum_{\pi \in \Pi} \alpha(\pi) y^t(\pi)$. By part (b) of Theorem 5.2.4, y^t is in the core of (N, f^t) for each t in T. By part (c) of Theorem 5.2.7, y^t is increasing in t on T. Therefore, if $t' \preceq t''$ ($t'' \preceq t'$), then $y^{t''}$ is in the core of $(N, f^{t''})$ and $y^{t'} \preceq y^{t''}$ ($y^{t''} \preceq y^{t'}$). Ichiishi [1990] characterizes pairs of cooperative games such that any payoff vector in the core of one cooperative game has an upper (or a lower) bound in the core of the other cooperative game.

A cooperative game (N, f) has a **population monotonic allocation scheme** if there exist $y_i^{N'}$ for each subset N' of the players N and each i in N' such that $\sum_{i \in N'} y_i^{N'} = f(N')$ and such that $y_i^{N'} \le y_i^{N''}$ for all subsets of players N' and N'' with N' being a subset of N'' and for each player i in N'. The payoff vector $\{y_i^{N'} : i \in N'\}$ is in the core of the subgame (N', f) for each subset N' of N. A population monotonic allocation scheme gives a payoff vector in the core of each subgame (N', f) such that each component of the payoff vector for the subgame (N', f) increases with N'; that is, the payoff to each player for each subgame in which the player participates increases with the set of players participating in the subgame. Part (a) of Theorem 5.2.8, given in Sprumont [1990] and attributed to T. Ichiishi, shows that the vectors generated by the greedy algorithm with any given permutation of the players yield a population monotonic allocation scheme for any convex game. Part (b) of Theorem 5.2.8, from Rosenthal [1990], shows that the Shapley value is a population monotonic allocation scheme for any convex game.

Theorem 5.2.8. *Suppose that* (N, f) *is a convex game.*
(a) *If* π *is any permutation of the players* N *and* $y(\pi)^{N'}_{\pi(j)} = f(S(\pi, j) \cap N') - f(S(\pi, j-1) \cap N')$ *for each subset* N' *of the players* N *and each player* $\pi(j)$ *in* N' *for* $j = 1, \ldots, n$*, then* $y(\pi)^{N'}_{\pi(j)}$ *is a population monotonic allocation scheme.*
(b) *The Shapley value for the subgame* (N', f)*, with* N' *being an arbitrary subset of* N*, is a population monotonic allocation scheme.*

Proof. Pick any permutation π of N. By part (a) of Lemma 5.2.1, $\sum_{i \in N'} y(\pi)^{N'}_i = f(N')$ for each subset N' of N. By the supermodularity of the characteristic function, $y(\pi)^{N'}_i$ is increasing in N' for subsets N' of N with i in N'. Therefore, $y(\pi)^{N'}_{\pi(j)}$ is a population monotonic allocation scheme and part (a) holds.

Pick any distinct subsets N' and N'' of the players N with N' being a subset of N''. Let Π' be the collection of all $|N'|!$ permutations of N' and let Π'' be the collection of all $|N''|!$ permutations of N''. For any permutation π' of N', let $y(\pi')^{N'}$ be the payoff vector generated by the greedy algorithm with permutation π' for the convex game (N', f). For any permutation π'' of N'', let $y(\pi'')^{N''}$ be the payoff vector generated by the greedy algorithm with permutation π'' for the convex game (N'', f). Let $y^{N'}$ be the Shapley value for the convex game (N', f) and let $y^{N''}$ be the Shapley value for the convex game (N'', f). For any permutation π' of N', there are exactly $|N''|!/|N'|!$ corresponding permutations π'' with the property that the players of N' appear in the same order in $\pi'(1), \ldots, \pi'(|N'|)$ and in $\pi''(1), \ldots, \pi''(|N''|)$. For any permutation π' of N', any of these $|N''|!/|N'|!$ corresponding permutations π'' of N'', and any player i in N', part (a) implies that $y(\pi')^{N'}_i \le y(\pi'')^{N''}_i$. Then

$$y^{N'}_i = (1/|N'|!) \textstyle\sum_{\pi' \in \Pi'} y(\pi')^{N'}_i \le (1/|N''|!) \sum_{\pi'' \in \Pi''} y(\pi'')^{N''}_i = y^{N''}_i$$

for each i in N', and part (b) holds. □

5.2.3 Structure of the Core

Notwithstanding the many strong properties known for the core of a convex game, the issue of **bounded rationality** (that is, the hypothesis that a real decision maker has limited computational resources available) may be significant because of the large number of linear inequality constraints, 2^n for a cooperative game with n players, that define the core. As an illustration, consider the problem of a decision maker who has the responsibility of proposing a payoff vector to the players in a cooperative game. Suppose that the decision maker has found a possible payoff vector whose components sum to the return $f(N)$ attainable by the coalition N of all players (so the payoff vector is feasible) and that has properties making it desirable to the decision maker if only it were acceptable to all the players. The decision maker must determine if the vector is a member of the core, because otherwise some strict subset of the players would choose to form a coalition and operate independently rather than accept the payoffs. A direct approach would involve checking each of the 2^n inequalities defining the core. Yet if there are only 40 players, the number of inequalities exceeds 10^{12}. Even though the direct approach to determining whether or not the payoff vector is in the core is simple and straightforward, its computational complexity could preclude making this determination in a practical time frame. Deng and Papadimitriou [1994] study the computational complexity of problems related to standard solution concepts for cooperative games with side payments, and, using a version of the trading game considered in Subsection 5.3.4, they find that certain problems involving the core can be handled efficiently when the trading game is a convex game but that these problems are generally intractable when the trading game need not be a convex game.

There are three classes of problems related to the core where computation complexity could be an issue. The first involves the selection of a certain payoff vector in the core. The second issue involves the validation that a particular payoff vector is or is not in the core (or in a specified subset of the core). The third involves the determination of whether or not a particular payoff vector is in the core. (The complexity of validating membership may be simpler than the complexity of determining membership, because validation may rely on particulars not available for the determination of membership as discussed below.) These complexity issues sharpen the traditional focus (Sharkey [1982c]) on the existence of a nonempty core as an indicator of stability.

One measure of computational efficiency, that the complexity is **strongly polynomial**, is that the time required to solve any instance of the problem is bounded above by a function that is polynomial in the dimension n of the problem and does not otherwise depend on any other parameter of the problem. A weaker measure of computational efficiency, that the complexity is **polynomial**, is that the time required to solve any instance of the problem is bounded above by a function that is polynomial in the size of the specification of the data input into a problem. The time required to solve an arbitrary instance of a particular class of problems of "size" no greater than k is $\boldsymbol{O(g(k))}$ for some nonnegative function $g(k)$ of k if there exists a positive constant γ not depending on the problem such that the solution time for all such instances is bounded above by $\gamma g(k)$. See Garey and Johnson [1979] and Papadimitriou [1994] for a detailed and extensive discussion of computational complexity. Assume throughout this subsection that each evaluation of the characteristic function and each arithmetic operation can be done in time bounded above by a constant.

Selecting an arbitrary member of the core of a convex game (or an arbitrary extreme point of the core) can be accomplished by picking any permutation π of the players N and generating $y(\pi)$ by the greedy algorithm with the permutation π. The complexity of implementing the greedy algorithm is $O(n)$. The resulting payoff vector $y(\pi)$ is a member of the core by part (a) of Theorem 5.2.1, and, indeed, is an extreme point of the core by part (a) of Theorem 5.2.4. Furthermore, one can use the greedy algorithm as in Theorem 5.2.2 to find a payoff vector in the core of a convex game that lexicographically maximizes a vector of m linear functions over all payoff vectors in the core. For this latter problem, the total complexity is dominated by the $O(mn\log_2(n))$ complexity of lexicographically ordering the m-vectors of coefficients of the n variables in order to yield the appropriate permutation with which to implement the $O(n)$ greedy algorithm.

The natural way to examine the complexity of validating core membership is to first consider validating that a payoff vector is an extreme point of the core and then to consider validating that a payoff vector is a member of the core. By part (a) of Theorem 5.2.4, one can validate that a payoff vector y' is an extreme point of the core of a convex game by providing a permutation π of the players and then demonstrating that the payoff vector y' is generated by the greedy algorithm with the permutation π; that is, that $y' = y(\pi)$. The complexity of this validation is $O(n)$. (Note that the complexity of finding this permutation π is not included in the complexity of validating core membership.) Alternatively, if a payoff vector y' is not an extreme point of the core, then this can be validated by providing a strict subset S' of the players N with $f(S') = \sum_{i \in S'} y'_i$ and demonstrating in time $O(n)$ that $f(S' \cup \{i'\}) \neq \sum_{i \in S' \cup \{i'\}} y'_i$ for each i' in $N \setminus S'$. (See Algorithm 5.2.1, Theorem 5.2.11, and the second paragraph of the proof of Theorem 5.2.11.) By part (b) of Theorem 5.2.4, one can validate that a payoff vector y' is in the core by providing permutations $\pi^1, \ldots, \pi^{n+1}$ of the players N and nonnegative real numbers $\alpha_1, \ldots, \alpha_{n+1}$ with $\sum_{j=1}^{n+1} \alpha_j = 1$ and then demonstrating that $y' = \sum_{j=1}^{n+1} \alpha_j y(\pi^j)$. (Recall that $y(\pi^j)$ is the extreme point of the core generated by the greedy algorithm with permutation π^j.) The complexity of this validation is $O(n^2)$, which includes generating each of the $n+1$ extreme points $y(\pi^j)$ with the greedy algorithm and the other arithmetic operations. Alternatively, if a payoff vector y' is not in the core, then this can be validated either by observing that $\sum_{i \in N} y'_i \neq f(N)$ or by providing a subset S' of the players for which the corresponding core constraint is violated and then demonstrating in time $O(n)$ that $f(S') > \sum_{i \in S'} y'_i$.

Before proceeding to consider the problem of determining membership in the core of a convex game, some further clarification of distinctions between the problems of validating membership and of determining membership are in order. The complexity of validation stems from the intrinsic difficulty of presenting a proof confirming that a particular payoff vector is or is not a member of the core (or has or does not have some other specified property, like being a member of the set of extreme points of the core). However, the complexity of the validation problem does not take into account the complexity of identifying information given as an input for the proof. The complexity of validating that a particular payoff vector is an extreme point of the core of a convex game is $O(n)$, as noted above. But this validation takes as input a particular permutation of the players, and it is seen below that the complexity of determining membership in the set of extreme points of the core of a convex game and the complexity of computing the permutation input for the corresponding validation problem is $O(n^2)$ by Algorithm 5.2.1 and Theorem 5.2.11. Likewise, it is noted above that the complexity of validating that a particular payoff vector is in the core of a convex game is $O(n^2)$, while discussion below reveals that much less can be said about the complexity of the corresponding problem of determining membership. Here, the validation takes as input $n + 1$ particular permutations of the players and $n + 1$ particular real numbers, while the corresponding problem of determining membership starts from scratch and may well proceed by computing these permutations and real numbers.

A fundamental problem is that of determining in a computationally efficient manner whether or not a given payoff vector is in the core of a convex game. (The same membership problem exists for polymatroids.) There are four special cases for which efficient combinatorial algorithms (that is, algorithms that are strongly polynomial) are known for determining membership in the core of a convex game or in a polymatroid. One is an algorithm of Topkis [1983] for determining membership in the core of an activity selection game, as described in Subsection 5.5.4. (Deng and Papadimitriou [1994] give a version of this efficient algorithm for determining membership in a trading game, as discussed in Subsection 5.3.4, which is a special case of an activity selection game.) The second is an algorithm of Cunningham [1984] for determining membership in a matroid polyhedron, where a **matroid polyhedron** is a polymatroid for which each component of each extreme point is 0 or 1. The others involve two network synthesis games studied by Tamir [1991].

Given a convex game (N, f) and a payoff vector y' with $f(N) = \sum_{i\in N} y'_i$, let $g(S) = f(S) - \sum_{i\in S} y'_i$ for each subset S of N. By Theorem 2.6.4 and part (b) of Lemma 2.6.1, $g(S)$ is supermodular. If S' maximizes $g(S)$ over all subsets

S of N, then $g(S') \le 0$ implies that y' is in the core of (N, f) and $g(S') > 0$ implies that $f(S') > \sum_{i \in S'} y'_i$ and y' is not in the core of (N, f). Hence, any method for maximizing a supermodular function over all subsets of a given finite set N can be used to determine whether a particular payoff vector is in the core of a convex game with the set of players N. Grötschel, Lovász, and Schrijver [1981] give an ellipsoid method that can maximize a supermodular function $g(S)$ over all subsets S of a finite set N in polynomial time and hence can be used to determine in polynomial time whether or not a given payoff vector is in the core of a convex game. A strongly polynomial method for solving these problems is not known.

The present subsection develops combinatorial structural properties of the core of a convex game and is motivated, in part, by the goal of providing insights that may shed some light on the unresolved issue of efficiently determining whether or not a given payoff vector is in the core of a convex game. The progression of results looks into properties of a single extreme point of the core of a convex game (Theorem 5.2.9, Theorem 5.2.10, Algorithm 5.2.1, and Theorem 5.2.11), then explores properties of two adjacent extreme points of the core of a convex game (Theorem 5.2.12), and finally considers properties of some or all paths that are strictly increasing (as defined below) with respect to a given linear objective function on the core of a convex game (Theorem 5.2.13.).

Definitions in this paragraph are with respect to a convex game (N, f). For a payoff vector y, a subset S of the players N is ***y*-tight** if $\sum_{i \in S} y_i = f(S)$. If y is in the core of (N, f), then the y-tight sets are those that maximize the supermodular function $f(S) - \sum_{i \in S} y_i$ over all subsets S of N and so, by Theorem 2.7.1, the collection of y-tight sets is a sublattice of $\mathcal{P}(N)$; that is, the union and the intersection of any two y-tight sets are also y-tight. (Conversely, as in Bixby, Cunningham and Topkis [1985] and Monma and Topkis [1982], for any finite set N and any sublattice Ψ of $\mathcal{P}(N)$ that includes $\emptyset$ and N, there is a supermodular function $f(S)$ on $\mathcal{P}(N)$ with $f(\emptyset) = 0$ and a payoff vector y in the core of (N, f) such that the collection of all y-tight sets is Ψ. To confirm this, construct the characteristic function $f(S)$ as $f(S) = \max\{|Z| : Z \in \Psi, Z \subseteq S\}$ and let each component of the payoff vector y be 1 so $f(S)$ is supermodular by Theorem 2.7.6, y is in the core of (N, f), and the collection of y-tight sets in (N, f) is Ψ. See Theorem 3.7.7.) For each payoff vector y and each player i in N, let $B(y, i)$ be the intersection of all y-tight sets containing i; that is,

$$B(y, i) = \cap\{S : S \subseteq N, i \in S, \textstyle\sum_{i \in S} y_i = f(S)\}.$$

(As a convention, define the intersection and the union of an empty collection of sets to be the empty set.) If $B(y, i)$ is nonempty, then i is in $B(y, i)$. If y is in the core, then $B(y, i)$ is nonempty for each i in N. If y is in the core of (N, f), then $B(y, i)$ is y-tight for each i in N because the intersection of y-tight sets is y-tight. For any payoff vector y, define the binary relation $\preceq^y$ on pairs of elements of N such that $i' \preceq^y i''$ for i' and i'' in N if $B(y, i')$ is a subset of $B(y, i'')$; observe that when $B(y, i')$ is nonempty, as is the case for y in the core of (N, f), $B(y, i')$ is a subset of $B(y, i'')$ if and only if i' is in $B(y, i'')$. For any payoff vector y and any i in N, let $C(y, i)$ be those elements of N that are covered by i with respect to $\preceq^y$; that is, i' is in $C(y, i)$ if and only if i' is in $B(y, i)$ and there is no i'' in $B(y, i) \setminus \{i\}$ with i' in $B(y, i'') \setminus \{i''\}$. If the set of players N is a partially ordered set with an ordering relation $\preceq$, then a permutation π of N is **compatible** with the ordering relation $\preceq$ if $\pi(j') \prec \pi(j'')$ for j' and j'' with $1 \leq j' \leq n$ and $1 \leq j'' \leq n$ implies that $j' < j''$. (If N is a partially ordered set with the ordering relation $\preceq$, then a permutation π compatible with $\preceq$ can be constructed as follows. Let $\pi(1)$ be a minimal element of N with respect to $\preceq$. Given $\pi(1), \ldots, \pi(j-1)$ for any integer j with $2 \leq j \leq n$, let $\pi(j)$ be a minimal element of $N \setminus \{\pi(1), \ldots, \pi(j-1)\}$ with respect to $\preceq$.) If π is a permutation of the players N and i' and i'' are distinct players in N, let $\tau(\pi, i', i'')$ denote the permutation of N that is identical to π except that the positions of i' and i'' in the permutation are exchanged. If π is a permutation of the players N, $y = y(\pi)$ is generated by the greedy algorithm with the permutation π, j is an integer with $2 \leq j \leq n$, $\pi(j-1) = i'$, and $\pi(j) = i''$, then the payoff vector $y(\tau(\pi, i', i''))$ generated by the greedy algorithm with the permutation $\tau(\pi, i', i'')$ is identical to y except that $y(\tau(\pi, i', i''))_{i'} = y_{i'} + f(S(\pi, j-2) \cup \{i', i''\}) - f(S(\pi, j-2) \cup \{i''\}) - f(S(\pi, j-2) \cup \{i'\}) + f(S(\pi, j-2))$ and $y(\tau(\pi, i', i''))_{i''} = y_{i''} - f(S(\pi, j-2) \cup \{i', i''\}) + f(S(\pi, j-2) \cup \{i'\}) + f(S(\pi, j-2) \cup \{i''\}) - f(S(\pi, j-2))$.

Theorem 5.2.9, from Bixby, Cunningham, and Topkis [1985], characterizes the extreme points of the core of a convex game as those payoff vectors y in the core for which the set of players is a partially ordered set with the ordering relation $\preceq^y$, characterizes those permutations with which the greedy algorithm generates a given extreme point of the core, and shows that the ordering relation for each extreme point yields a chain if and only if the characteristic function is strictly supermodular.

Theorem 5.2.9. *Suppose that (N, f) is a convex game.*
(a) *If y is an extreme point of the core, then the set of players N is a partially ordered set with the ordering relation $\preceq^y$.*

(b) *If y is in the core and the set of players N is a partially ordered set with the ordering relation $\preceq^y$, then y is an extreme point of the core.*
(c) *If y is an extreme point of the core and the permutation π of the players N is compatible with $\preceq^y$, then the greedy algorithm generates y with permutation π; that is, $y = y(\pi)$.*
(d) *If y is an extreme point of the core and the permutation π of the players N is such that the greedy algorithm generates y with permutation π (that is, $y = y(\pi)$), then the permutation π is compatible with $\preceq^y$.*
(e) *If $f(S)$ is strictly supermodular and y is an extreme point of the core, then the set of players N is a chain with the ordering relation $\preceq^y$.*
(f) *If the set of players N is a chain with the ordering relation $\preceq^y$ for each extreme point y of the core, then $f(S)$ is strictly supermodular.*

Proof. Suppose that y is an extreme point of the core of (N, f). The binary relation $\preceq^y$ is reflexive because $B(y, i) \subseteq B(y, i)$ and hence $i \preceq^y i$ for each i in N. The binary relation $\preceq^y$ is transitive because if i', i'', and i''' in N are such that $i' \preceq^y i''$ and $i'' \preceq^y i'''$, then $B(y, i') \subseteq B(y, i'')$ and $B(y, i'') \subseteq B(y, i''')$, implying that $B(y, i') \subseteq B(y, i''')$ and hence $i' \preceq^y i'''$. Let i' and i'' be any two distinct elements of N. Suppose that $i' \preceq^y i''$ and $i'' \preceq^y i'$. Then each y-tight set containing i' also contains i'' and each y-tight set containing i'' also contains i'. Therefore, there exists $\epsilon > 0$ such that $y + \epsilon u^{i'} - \epsilon u^{i''}$ and $y - \epsilon u^{i'} + \epsilon u^{i''}$ are in the core of (N, f). But this contradicts y being an extreme point of the core because $y = (1/2)(y + \epsilon u^{i'} - \epsilon u^{i''}) + (1/2)(y - \epsilon u^{i'} + \epsilon u^{i''})$. Hence, the binary relation $\preceq^y$ is antisymmetric. Consequently, N is a partial ordered set and part (a) holds.

Now suppose that y is in the core and N is a partially ordered set with the ordering relation $\preceq^y$. Let π be any permutation of N that is compatible with $\preceq^y$. Then $B(y, \pi(j))$ is a nonempty subset of $S(\pi, j)$ for $j = 1, \ldots, n$. Therefore $\cup_{j=1}^k B(y, \pi(j)) = S(\pi, k)$ for $k = 0, \ldots, n$. Because $B(y, i)$ is a y-tight set for each i and the union of y-tight sets is y-tight, $S(\pi, j)$ is a y-tight set for $j = 1, \ldots, n$. Then for $j = 1, \ldots, n$,

$$y_{\pi(j)} = \textstyle\sum_{i \in S(\pi, j)} y_i - \sum_{i \in S(\pi, j-1)} y_i = f(S(\pi, j)) - f(S(\pi, j-1)).$$

Thus $y = y(\pi)$ and y is generated by the greedy algorithm with permutation π. By part (a) of Theorem 5.2.4, y is an extreme point of the core and part (b) holds.

Suppose that y is an extreme point of the core and the permutation π is compatible with $\preceq^y$. By part (a), N is a partially ordered set with the ordering relation $\preceq^y$. The proof of part (b) shows that $y = y(\pi)$ and y is generated by the greedy algorithm with permutation π, so part (c) holds.

Suppose that y is an extreme point of the core and the greedy algorithm generates y with permutation π. For $j = 1, \ldots, n$, $S(\pi, j-1)$ is y-tight by part (a) of Lemma 5.2.1, $S(\pi, j-1)$ includes $\{y_{\pi(1)}, \ldots, y_{\pi(j-1)}\}$, and $S(\pi, j-1)$ does not include $y_{\pi(j)}$. If j' and j'' are integers with $1 \leq j' < j'' \leq n$, then $\pi(j'')$ is not in $B(y, \pi(j'))$ and so π is compatible with $\preceq^y$ and part (d) holds.

Suppose that $f(S)$ is strictly supermodular. Let y be any extreme point of the core. By part (c), the greedy algorithm generates y with any permutation compatible with $\preceq^y$. By Lemma 5.2.2, there is exactly one permutation with which the greedy algorithm generates y. Therefore, there is only one permutation compatible with $\preceq^y$. Hence, N is a chain with the ordering relation $\preceq^y$, and part (e) holds.

Finally, suppose that N is a chain with the ordering relation for each extreme point of the core. Then for each extreme point y of the core there is only one permutation of N compatible with $\preceq^y$ and by part (d) there is only one permutation with which the greedy algorithm generates y. Then $f(S)$ is strictly supermodular by Lemma 5.2.2, and part (f) holds. □

If (N, f) is a convex game, y is an extreme point of the core, and i' and i'' are players in N with i' in $C(y, i'')$, then part (c) of Theorem 5.2.9 implies that there exists a permutation π of N such that $S(\pi, |B(y, i'')|) = B(y, i'')$, $\pi(|B(y, i'')|) = i''$, $\pi(|B(y, i'')| - 1) = i'$, and the greedy algorithm generates y with the permutation π. If (N, f) is a convex game and y' and y'' are extreme points of the core that differ in exactly two components i' and i'', then, by part (c) of Theorem 5.2.9, i' is in $C(y', i'')$ if and only if there exists a permutation π of N such that the greedy algorithm generates y' with the permutation π and i' appears directly before i'' in π; furthermore, if such a permutation π exists, then $y'_{i'} < y''_{i'}$, y'' is generated by the greedy algorithm with the permutation $\tau(\pi, i', i'')$, and i'' is in $C(y'', i')$.

Theorem 5.2.10, from Bixby, Cunningham, and Topkis [1985], characterizes those extreme points of the core of a convex game that maximize a given linear function over the core (or, equivalently, characterizes those linear functions that are maximized over the core by a given extreme point) in terms of the ordering relation associated with each extreme point.

Theorem 5.2.10. *If (N, f) is a convex game, y' is an extreme point of the core, and r is in R^n, then y' maximizes $r \cdot y$ over all y in the core of (N, f) if and only if $r_{i'} \leq r_{i''}$ for all i' and i'' in N with $i' \preceq^{y'} i''$.*

Proof. Suppose that $r_{i'} \leq r_{i''}$ for all i' and i'' in N with $i' \preceq^{y'} i''$. Then there exists a permutation π of N that is compatible with $\preceq^{y'}$ and for which $r_{\pi(j)}$ is increasing in j for $j = 1, \ldots, n$. By part (c) of Theorem 5.2.9, $y' = y(\pi)$. By Theorem 5.2.2, y' maximizes $r \cdot y$ over y in the core.

Now suppose that y' maximizes $r \cdot y$ over y in the core. Suppose also that there exist i' and i'' in N with $i' \preceq^{y'} i''$ and $r_{i'} > r_{i''}$. Then there exist k' and k'' in N with k' in $C(y', k'')$ and $r_{k'} > r_{k''}$, and so there is a permutation π of N such that the greedy algorithm generates y' with the permutation π and k' directly precedes k'' in π. By part (d) of Theorem 5.2.9, the payoff vector $y(\tau(\pi, k', k''))$ generated by the greedy algorithm with the permutation $\tau(\pi, k', k'')$ is distinct from y'. Therefore, $y(\tau(\pi, k', k''))$ and y' differ only in components k' and k'' with $y(\tau(\pi, k', k''))_{k'} = y'_{k'} + \epsilon$ and $y(\tau(\pi, k', k''))_{k''} = y'_{k''} - \epsilon$ for some $\epsilon > 0$. This together with $r_{k'} > r_{k''}$ contradicts the optimality of y', and so $r_{i'} \leq r_{i''}$ for all i' and i'' in N with $i' \preceq^{y'} i''$. □

Algorithm 5.2.1 determines whether or not a given payoff vector y is an extreme point of the core of a convex game (N, f). If so, Algorithm 5.2.1 also gives a permutation π with which the greedy algorithm generates y and the sets $B(y, i)$ for each player i that define the ordering relation $\preceq^y$ on N. The algorithm proceeds to myopically construct the permutation π element by element, if possible. Given $\pi(1), \ldots, \pi(j-1)$ for any integer j with $1 \leq j \leq n$, the algorithm selects $\pi(j)$ as any element of $N \setminus \{\pi(1), \ldots, \pi(j-1)\}$ such that $\{\pi(1), \ldots, \pi(j)\}$ is y-tight. If Algorithm 5.2.1 is successful in constructing a permutation $\pi = (\pi(1), \ldots, \pi(n))$ of N, then the greedy algorithm generates y with the permutation π and y is an extreme point of the core. Otherwise, y is not an extreme point of the core. Algorithm 5.2.1 can be implemented so that it runs in time $O(n^2)$. Theorem 5.2.11 establishes the validity of the output of Algorithm 5.2.1. Algorithm 5.2.1 and Theorem 5.2.11 are from Bixby, Cunningham, and Topkis [1985].

Algorithm 5.2.1. Given a convex game (N, f) and a payoff vector y, proceed as follows to determine whether y is an extreme point of the core, and, if so, to construct the sets $B(y, i)$ for each player i in N and the permutation π.
(a) Set $j = 0$ and $D = \emptyset$.
(b) Set $j = j + 1$. If there is any i' in $N \setminus D$ with $f(D \cup \{i'\}) = \sum_{i \in D \cup \{i'\}} y_i$, then pick any such i', set $D = D \cup \{i'\}$, set $\pi(j) = i'$, and continue to step (c). Otherwise, stop.
(c) Set $B_j = D$ and set $k = j - 1$.
(d) If $f(B_j \setminus \{\pi(k)\}) = \sum_{i \in B_j \setminus \{\pi(k)\}} y_i$, set $B_j = B_j \setminus \{\pi(k)\}$. Set $k = k - 1$. If $k = 0$, then go to step (b) and continue. Otherwise, repeat this step.

Theorem 5.2.11. *Suppose that a convex game (N, f) and a payoff vector y are given. Algorithm* 5.2.1 *terminates in some iteration of step* (b). *If $D \neq N$ at termination, then y is not an extreme point of the core. If $D = N$ at termination, then y is an extreme point of the core, $B(y, \pi(j)) = B_j$ for $j = 1, \ldots, n$, and y is generated by the greedy algorithm with the permutation π.*

Proof. There is at most one iteration of step (a). There are at most n iterations of step (b), at most n iterations of step (c), and at most $n(n-1)/2$ iterations of step (d). Therefore, Algorithm 5.2.1 must terminate at some iteration of step (b).

Suppose that $D \neq N$ at termination and that y is an extreme point of the core. By part (a) of Theorem 5.2.9, N is a partially ordered set with the ordering relation $\preceq^y$. Let i' be a minimal element of $N \setminus D$ relative to $\preceq^y$. By the choice of i', $B(y, i')$ is a subset of $D \cup \{i'\}$ and so $D \cup \{i'\} = D \cup B(y, i')$. The set D is y-tight by the construction of step (b), and $B(y, i')$ is y-tight because it is the intersection of y-tight sets. Therefore, $D \cup \{i'\}$ is y-tight because it is the union of y-tight sets. But $D \cup \{i'\}$ being y-tight contradicts the supposition that Algorithm 5.2.1 terminates in step (b) with i' not in D. Therefore, $D \neq N$ at termination implies that y is not an extreme point of the core.

Now suppose that $D = N$ at termination. By the construction of step (b), the set D is y-tight after each iteration of step (b) and so $\{\pi(1), \ldots, \pi(j)\}$ is y-tight for $j = 0, \ldots, n$. This implies that y is generated by the greedy algorithm with permutation π, so y is an extreme point of the core by part (a) of Theorem 5.2.4. By the constructions of step (b) and step (d), B_j is y-tight for $j = 1, \ldots, n$. Because $\pi(j)$ is in the y-tight set B_j, $B(y, \pi(j))$ is a subset of B_j for each j. Suppose that $B(y, \pi(j')) \neq B_{j'}$ for some j'. Let k' be such that $\pi(k')$ is a maximal element of $B_{j'} \setminus B(y, \pi(j'))$ relative to $\preceq^y$. Then $\pi(k')$ is a maximal element of $B_{j'}$ relative to $\preceq^y$. Let $B'_{j'}$ be the set $B_{j'}$ at the beginning of the iteration of step (d) for $j = j'$ where Algorithm 5.2.1 considers whether to remove $\pi(k')$ from $B_{j'}$ by determining whether $B_{j'} \setminus \{\pi(k')\}$ is y-tight. (Distinguishing this $B'_{j'}$ from the final $B_{j'}$ is needed because the set $B_{j'}$ may change through the iterations of step (d) given $j = j'$.) By step (d) of Algorithm 5.2.1, $B_{j'}$ is a subset of $B'_{j'}$ and $B'_{j'} \setminus B_{j'}$ is a subset of $\{\pi(1), \ldots, \pi(k'-1)\}$. By part (d) of Theorem 5.2.9, $\pi(k')$ is a maximal element of $(B'_{j'} \setminus B_{j'}) \cup \{\pi(k')\}$ relative to $\preceq^y$ and so $\pi(k')$ is a maximal element of $B'_{j'} = B_{j'} \cup (B'_{j'} \setminus B_{j'})$ relative to $\preceq^y$. Because $B'_{j'}$ is y-tight, $B(y, i)$ is a subset of $B'_{j'}$ for each i in $B'_{j'}$ and so $\cup_{i \in B'_{j'} \setminus \{\pi(k')\}} B(y, i)$ contains $B'_{j'} \setminus \{\pi(k')\}$ and is a subset of $B'_{j'}$. Because $\pi(k')$ is a maximal element of $B'_{j'}$ relative to $\preceq^y$, $B'_{j'} \setminus \{\pi(k')\} = \cup_{i \in B'_{j'} \setminus \{\pi(k')\}} B(y, i)$ and so $B'_{j'} \setminus \{\pi(k')\}$ is the union of y-tight sets and hence is y-tight. But this would imply that $\pi(k')$ would be removed from $B_{j'}$ in step (d), which is a contradiction, so $B(y, \pi(j)) = B_j$ for each j. □

A subset of R^n that is the set of solutions of a finite system of linear inequalities is a **polyhedron**. Two distinct extreme points x' and x'' of a polyhedron P are **adjacent** if each point on the line segment joining x' and x'' is not contained in any line segment between two points of P that are not on the line segment joining x' and x''. Equivalently, two distinct extreme points x' and x'' of a polyhedron P in R^n are adjacent if the collection of vectors of coefficients of those linear inequalities determining P that hold as equalities for both x' and x'' has rank $n-1$.

Theorem 5.2.12, from Topkis [1984a], characterizes the adjacency relation among extreme points of the core of a convex game by showing that two extreme points are adjacent if and only if the extreme points differ in exactly two components and one of these components covers the other with respect to the ordering relation associated with one of the extreme points.

Theorem 5.2.12. *If y' and y'' are extreme points of the core of a convex game (N, f), then y' and y'' are adjacent if and only if they differ in exactly two components, say i' and i'', and i' is in $C(y', i'')$. If this is the case, then $y''_{i'} = y'_{i'} + \epsilon$ and $y''_{i''} = y'_{i''} - \epsilon$ where $\epsilon = f(B(y', i'')) - f(B(y', i'') \setminus \{i'\}) - y'_{i'} > 0$.*

Proof. Suppose that y' and y'' differ in exactly two components, i' and i'', where i' is in $C(y', i'')$. Let $j' = |B(y', i'')|$. By part (c) of Theorem 5.2.9, there exists a permutation π such that the greedy algorithm generates y' with the permutation π, $S(\pi, j') = B(y', i'')$, $\pi(j') = i''$, and $\pi(j'-1) = i'$. Observe that $\tau(\pi, i', i'')(j') = i'$ and $\tau(\pi, i', i'')(j'-1) = i''$. The payoff vector $y(\tau(\pi, i', i''))$ generated by the greedy algorithm with permutation $\tau(\pi, i', i'')$ has $y(\tau(\pi, i', i''))_i = y'_i$ for each i with $i \neq i'$ and $i \neq i''$. Furthermore, $y(\tau(\pi, i', i'')) \neq y'$ by part (d) of Theorem 5.2.9 because i' is in $C(y', i'')$. Thus, y'' is generated by the greedy algorithm with the permutation $\tau(\pi, i', i'')$. The $n-1$ sets $S(\pi, j)$ for j in $\{1, \ldots, n\} \setminus \{j'-1\}$ are y'-tight and y''-tight, so y' and y'' are adjacent. Because the greedy algorithm generates y' with permutation π and generates y'' with permutation $\tau(\pi, i', i'')$,

$$
\begin{aligned}
y'_{i''} - y''_{i''} = y''_{i'} - y'_{i'}& \\
&= f(S(\tau(\pi, i', i''), j')) - f(S(\tau(\pi, i', i''), j'-1)) - y'_{i'} \\
&= f(S(\pi, j')) - f(S(\pi, j') \setminus \{i'\}) - y'_{i'} \\
&= f(B(y', i'')) - f(B(y', i'') \setminus \{i'\}) - y'_{i'} \\
&> 0.
\end{aligned}
$$

Now suppose that y' and y'' are adjacent. If for each pair i' and i'' of distinct elements in N there exists a set that is both y'-tight and y''-tight and contains exactly one of i' and i'', then the inequalities that are both y'-tight and y''-tight would have a unique solution, which is a contradiction. Thus there exist distinct elements i' and i'' of N such that each set that is both y'-tight and y''-tight either contains both i' and i'' or contains neither i' nor i''. For any real

ϵ, define the payoff vector y^ϵ to be identical to y' in all components except $y^\epsilon_{i'} = y'_{i'} + \epsilon$ and $y^\epsilon_{i''} = y'_{i''} - \epsilon$. Then each set that is both y'-tight and y''-tight is also y^ϵ-tight for each real ϵ. Because y' and y'' are adjacent, the collection of vectors of coefficients of those linear inequalities determining the core of (N, f) that correspond to sets that are both y'-tight and y''-tight has rank $n-1$. Then the set of vectors y such that all sets that are both y'-tight and y''-tight are also y-tight must be a line and this line must consist of the vectors y^ϵ for each real ϵ. Hence y' and y'' differ in exactly two components, i' and i''. Either $y'_{i'} < y''_{i'}$ or $y'_{i''} < y''_{i''}$. Without loss of generality, suppose that $y'_{i'} < y''_{i'}$. Then $y'_{i''} > y''_{i''}$. Because $B(y', i'')$ being y'-tight and y'' in the core imply

$$\textstyle\sum_{i\in B(y',i'')} y'_i = f(B(y', i'')) \leq \sum_{i\in B(y',i'')} y''_i,$$

it follows that i' is in $B(y', i'')$ and $B(y', i'')$ is y''-tight. Similarly,

$$\textstyle\sum_{i\in B(y'',i')} y''_i = f(B(y'', i')) \leq \sum_{i\in B(y'',i')} y'_i$$

implies that i'' is in $B(y'', i')$ and $B(y'', i')$ is y'-tight. Hence, $B(y', i'') = B(y'', i')$. Suppose that i' is not in $C(y', i'')$, so there exists i''' in $N \setminus \{i', i''\}$ with i' in $B(y', i''')$ and i''' in $B(y', i'') = B(y'', i')$. For any real ϵ, define the payoff vector z^ϵ to be identical to y' in all components except $z^\epsilon_{i'} = y'_{i'} + \epsilon$ and $z^\epsilon_{i'''} = y'_{i'''} - \epsilon$. Then i' in $B(y', i''')$ and i''' in $B(y'', i')$ imply that each set that is both y'-tight and y''-tight must either contain both i' and i''' or contain neither i' nor i''', so each such set is also z^ϵ-tight for each real ϵ. This is a contradiction. Therefore, i' is in $C(y', i'')$. □

By Theorem 5.2.12, each extreme point adjacent to a given extreme point of the core of a convex game (N, f) corresponds to a distinct pair of distinct elements from N and so an extreme point of the core of (N, f) can have no more that $n(n-1)/2$ adjacent extreme points. If (N, f) is a convex game and $f(S)$ is strictly supermodular, then N is a chain with the ordering relation of any extreme point of the core by part (e) of Theorem 5.2.9 and so for any extreme point of the core there are exactly $n-1$ ordered pairs of elements of N with the first of the pair covering the second of the pair with respect to the ordering relation of the extreme point and Theorem 5.2.12 implies that each extreme point of the core has exactly $n-1$ adjacent extreme points.

Suppose that (N, f) is a convex game, y' is an extreme point of the core, π' is a permutation of the players N such that y' is generated by the greedy algorithm with permutation π', and i' and i'' are players in N such that i' appears directly before i'' in the permutation π'. Let y'' be the extreme point of the core that is generated by the greedy algorithm with permutation $\tau(\pi, i', i'')$. Either i' is in $C(y', i'')$ or i' is not in $B(y', i'')$. If i' is in $C(y', i'')$, then $y'' \neq y'$

by part (d) of Theorem 5.2.9 and y' and y'' are adjacent extreme points by Theorem 5.2.12. If i' is not in $B(y', i'')$, then $y'' = y'$ by part (c) of Theorem 5.2.9. Thus, if one begins with an extreme point of the core and some permutation with which the greedy algorithm generates that extreme point, if one obtains a new permutation by exchanging two successive elements in the initial permutation, and if the greedy algorithm generates a new extreme point of the core with the new permutation, then the new extreme point is either adjacent to the original extreme point or identical to the original extreme point.

Let P be a polyhedron in R^n. If $x^1, \dots, x^m$ are extreme points of P with x^k adjacent to x^{k+1} for $k = 1, \dots, m-1$, then these m extreme points are a **path** from x^1 to x^m and the **path length** is $m - 1$. The **distance** $D(x', x'')$ between two extreme points x' and x'' of P is the minimum path length over all paths from x' to x''. The **diameter** of P is the maximum of the distance $D(x', x'')$ over all pairs of extreme points x' and x'' of P. If r is in R^n, $x^1, \dots, x^m$ is a path in P, and $r \cdot x^k \le r \cdot x^{k+1}$ ($r \cdot x^k < r \cdot x^{k+1}$) for $k = 1, \dots, m-1$, then the path is ***r*-increasing** (**strictly *r*-increasing**). If the components of r are distinct, then any r-increasing path on the core of a convex game is strictly r-increasing by Theorem 5.2.12.

Theorem 5.2.13, from Topkis [1992], gives a uniform upper bound on the length of any strictly r-increasing path on the core of a convex game when the characteristic function is integer-valued and increasing, establishes the existence of (almost) strictly r-increasing paths considerably better than the upper bound, gives improved results where the characteristic function is strictly supermodular, and provides a bound on the diameter of the core of a convex game with this bound attained when the characteristic function is strictly supermodular. These bounds are considerably stronger than bounds that are known or conjectured to hold on general polyhedra (Klee and Kleinschmidt [1987]; Klee and Minty [1972]). (In connection with the statement of part (b) of Theorem 5.2.13, note that there may not generally exist a *strictly* r-increasing path from an arbitrary extreme point to an arbitrary extreme point that maximizes $r \cdot y$ over y in the core. However, if the components of r are distinct, then there is a unique extreme point that maximizes $r \cdot y$ over all y in the core, and the statement of part (b) of Theorem 5.2.13 simplifies in this case.)

Theorem 5.2.13. *Suppose that (N, f) is a convex game and r is in R^n.*
(a) *If $f(S)$ is integer-valued for each subset S of N and is increasing in S, then any strictly r-increasing path on the core has length no greater that $(n-1)f(N)$.*
(b) *If y' and y'' are extreme points of the core and y'' maximizes $r \cdot y$ over all y in the core, then there exists an r-increasing path from y' to y'' on the core such that the path length is no greater than $n(n-1)/2$ and the segment of the path*

from y' to the first extreme point on the path that maximizes $r \cdot y$ over all y in the core is strictly r-increasing.
(c) *If $f(S)$ is strictly supermodular, then each strictly r-increasing path on the core has length no greater than $n(n-1)/2$. Furthermore, if the components of r are distinct, y' minimizes $r \cdot y$ over all y in the core, and y'' maximizes $r \cdot y$ over all y in the core, then each strictly r-increasing path from y' to y'' on the core has length equal to $n(n-1)/2$.*
(d) *The diameter of the core is no greater than $n(n-1)/2$. Furthermore, if $f(S)$ is strictly supermodular, then the diameter of the core equals $n(n-1)/2$.*

Proof. First, suppose that $f(S)$ is integer-valued for each subset S of N and is increasing in S. Let $q_1, \dots, q_k$ be the distinct values of $\{r_i : i \in N\}$ for some integer k with $1 \le k \le n$, where these values are ordered so that $q_1 < q_2 < \dots < q_k$. Define r' in R^n such that $r'_i = j$ if $r_i = q_j$. By the adjacency characterization of Theorem 5.2.12, a path is strictly r-increasing on the core if and only if it is strictly r'-increasing on the core. If y' and y'' are extreme points of the core with $r' \cdot y' < r' \cdot y''$, then $r' \cdot y' + 1 \le r' \cdot y''$ because each component of y' and y'' is an integer by part (a) of Theorem 5.2.4 and each component of r' is an integer. Therefore, the length of a strictly r'-increasing path on the core cannot exceed the difference between the maximum of $r' \cdot y$ over all y in the core and the minimum of $r' \cdot y$ over all y in the core. The maximum of $r' \cdot y$ over all y in the core is no greater than $kf(N)$, and the minimum of $r' \cdot y$ over all y in the core is no less than $f(N)$. Then the length of a strictly r'-increasing path on the core cannot exceed $kf(N) - f(N) \le (n-1)f(N)$, and part (a) holds.

Now, suppose that y' and y'' are extreme points of the core and y'' maximizes $r \cdot y$ over all y in the core. By part (a) of Theorem 5.2.4, there exist permutations π' and π'' of N such that y' is generated by the greedy algorithm with the permutation π' and y'' is generated by the greedy algorithm with the permutation π''. In particular, pick π'' so that $r_{\pi''(j)} \le r_{\pi''(j+1)}$ for $j = 1, \dots, n-1$. (Pick π'' to minimize $|\{j : 1 \le j \le n-1, r_{\pi(j)} > r_{\pi(j+1)}\}|$ over all permutations π with which the greedy algorithm generates y''. If $r_{\pi''(j')} > r_{\pi''(j'+1)}$ for some j' with $\pi''(j')$ not in $B(y'', \pi''(j'+1))$, then the greedy algorithm generates y'' with the permutation $\tau(\pi'', \pi''(j'), \pi''(j'+1))$ by part (c) of Theorem 5.2.9, which contradicts the choice of π''. If $r_{\pi''(j')} > r_{\pi''(j'+1)}$ for some j' with $\pi''(j')$ in $B(y'', \pi''(j'+1))$, then $y(\tau(\pi'', \pi''(j'), \pi''(j'+1))) \ne y''$ by part (d) of Theorem 5.2.9 and $r_{\pi''(j')} > r_{\pi''(j'+1)}$ implies that $r \cdot y'' < r \cdot y(\tau(\pi'', \pi''(j'), \pi''(j'+1)))$, which contradicts the choice of y''. Thus $r_{\pi''(j)} \le r_{\pi''(j+1)}$ for $j = 1, \dots, n-1$.) Set $y^1 = y'$ and $\pi^1 = \pi'$. Given any positive integer k with a corresponding extreme point y^k and a permutation π^k of N satisfying $y^k = y(\pi^k)$, if there exist elements i' and i'' of N with i' directly preceding i'' in π^k and with $r_{i'} > r_{i''}$ then pick any such i' and i'', let $\pi^{k+1} = \tau(\pi^k, i', i'')$, let $y^{k+1} = y(\pi^{k+1})$, reset k to $k+1$, and continue this process of generating new extreme points and permutations. At each iteration k of this process, either $y^{k+1} = y^k$ (if i' is not in $B(y^k, i'')$) or y^{k+1} and y^k are adjacent (if i' is in $B(y^k, i'')$) by Theorem 5.2.12 and $r \cdot y^k < r \cdot y^{k+1}$ by Theorem 5.2.12 and $r_{i'} > r_{i''}$. The preceding process must eventually terminate because there are finitely many extreme points and $|\{(j', j'') : 1 \le j' < j'' \le n, r_{\pi^k(j')} > r_{\pi^k(j'')}\}|$ decreases by 1 at each iteration k. If $k = k'$ at termination, then $y^1, \ldots, y^{k'}$ with consecutive repetitions deleted is a strictly r-increasing path from y' to $y^{k'}$. By construction, $r_{\pi^{k'}(j)} \le r_{\pi^{k'}(j+1)}$ for $j = 1, \ldots, n-1$ and so $y^{k'}$ maximizes $r \cdot y$ over all y in the core by Theorem 5.2.2. Now given any integer $k \ge k'$ with a corresponding extreme point y^k and a permutation π^k of N satisfying $y^k = y(\pi^k)$, if there exist elements i' and i'' of N with i' directly preceding i'' in π^k and with i'' preceding i' in π'' then pick any such i' and i'', let $\pi^{k+1} = \tau(\pi^k, i', i'')$, let $y^{k+1} = y(\pi^{k+1})$, reset k to $k+1$, and continue this process of generating new extreme points and permutations. At each iteration $k \ge k'$ of this process, i' and i'' are such that $r_{i'} = r_{i''}$ (because $r_{\pi''(j)} \le r_{\pi''(j+1)}$ and $r_{\pi^{k'}(j)} \le r_{\pi^{k'}(j+1)}$ for $j = 1, \ldots, n-1$), either $y^{k+1} = y^k$ (if i' is not in $B(y^k, i'')$) or y^{k+1} and y^k are adjacent (if i' is in $B(y^k, i'')$) by Theorem 5.2.12, and $r \cdot y^k = r \cdot y^{k+1}$ by Theorem 5.2.12 and $r_{i'} = r_{i''}$. The preceding process must eventually terminate because at each iteration the number of pairs from N that appear in a different order in π^k and in π'' is reduced by one. If $k = k''$ at termination, then $y^{k''} = y''$ and $y^1, \ldots, y^{k''}$ with consecutive repetitions deleted has path length no greater than $n(n-1)/2$ (because a given pair i' and i'' of elements from N can be exchanged in at most one iteration k). Hence, part (b) holds. This also establishes the first statement of part (d) because for any two extreme points y' and y'' one can pick r in R^n such that y'' maximizes $r \cdot y$ over all y in the core (first letting π be a permutation with $y'' = y(\pi)$, which exists by part (a) of Theorem 5.2.4, and then picking r with $r_{\pi(1)} \le \ldots \le r_{\pi(n)}$ and applying Theorem 5.2.2) and so part (b) implies that $D(y', y'') \le n(n-1)/2$.

Suppose that $f(S)$ is strictly supermodular. Let $y^1, \ldots, y^{k'}$ be any strictly r-increasing path on the core. By Lemma 5.2.2 and part (a) of Theorem 5.2.4, there is a unique permutation π^k of N for $k = 1, \ldots, k'$ such that the greedy algorithm generates y^k with the permutation π^k; that is, $y(\pi^k) = y^k$. For $k = 1, \ldots, k'-1$, $r \cdot y^k < r \cdot y^{k+1}$ and the adjacency characterization of Theorem 5.2.12 imply that there exist elements $i(k)'$ and $i(k)''$ in N such that $i(k)'$ appears directly before $i(k)''$ in π^k, $i(k)''$ appears directly before $i(k)'$ in π^{k+1}, the other $n-2$ components of π^k and π^{k+1} are identical, and $r_{i(k)''} < r_{i(k)'}$.

Then there is a unique pair of elements in N corresponding to each k for $k = 1, \ldots, k' - 1$ and so the length of the path is no greater than $n(n-1)/2$. Now suppose that, in addition, the components of r are distinct, y^1 minimizes $r \cdot y$ over all y in the core, and $y^{k'}$ maximizes $r \cdot y$ over all y in the core. By Theorem 5.2.2, $r_{\pi^1(1)} > r_{\pi^1(2)} > \ldots > r_{\pi^1(n)}$ and $r_{\pi^{k'}(1)} < r_{\pi^{k'}(2)} < \ldots < r_{\pi^{k'}(n)}$. Then for each pair of distinct elements in N, there must be some k with $1 \leq k \leq k' - 1$ such that the permutations π^k and π^{k+1} differ by the exchange of the positions of the elements of this pair and the path length $k' - 1$ is at least $n(n-1)/2$. Hence, part (c) holds.

Finally, suppose again that $f(S)$ is strictly supermodular. Let π' be any permutation of the players N and let the permutation π'' be such that $\pi''(j) = \pi'(n-j+1)$ for $j = 1, \ldots, n$. Then $D(y(\pi'), y(\pi'')) = n(n-1)/2$ by Theorem 5.2.12 and part (e) of Theorem 5.2.9, and the second statement of part (d) holds. □

5.3 Examples of Convex Games

This section presents four examples of convex games, so the theory of Section 5.2 applies to each of these examples. Subsection 5.3.1 models the (equivalent) cooperative cost game of a monopoly firm producing multiple products that are consumed by multiple consumers. Subsection 5.3.2 considers the monotonic surplus sharing problem for a set of agents whose return from acting cooperatively exceeds the sum of their individual returns when each agent acts independently. Subsection 5.3.3 models the (equivalent) cooperative cost game involving assigning aircraft landing fees to the flights using an airport, where different flights require runways of differing minimum cost. Subsection 5.3.4 examines a trading game among a set of countries, where various subsets of the countries receive a joint return if the entire subset is enabled to engage in mutual trade by being included in the same trading block.

5.3.1 Monopoly Firm

Consider a firm that produces a nonnegative amount of each of m products $j = 1, \ldots, m$, where x is the m-vector of production for the m products and $c(x)$ is the firm's cost function for a production vector x. Suppose that the firm is a monopoly and its monopoly production level is x'. There are n consumers of the firm's output, with the set of all consumers (players) being $N = \{1, \ldots, n\}$. Given that market prices for the products would yield a total demand equal to the monopoly production level x', the consumption of each consumer i is the nonnegative m-vector $x^{(i)}$ and $\sum_{i \in N} x^{(i)} = x'$. The firm sets prices for its products so that r_i is the revenue from the output $x^{(i)}$ demanded by each consumer i. The firm wants to determine the n-vector r of revenues from the n consumers, given the monopoly production level x', so that total revenue equals total cost (that is, $\sum_{i\in N} r_i = c(x')$) and no competing firm serving any subset of consumers could produce the demands of those consumers for less than the total revenue obtained from those consumers by the firm (that is, $\sum_{i\in S} r_i \leq c(\sum_{i\in S} x^{(i)})$ for each subset S of the consumers N). Let $f(S) = -c(\sum_{i\in S} x^{(i)})$ for each subset S of N. Then the firm wants $-r$ to be in the core of the cooperative game (N, f). Part (a) of Theorem 5.3.1, from Panzar and Willig [1977], shows that (N, f) is a convex game if the products consumed at a positive level by any two distinct consumers are distinct and if the production cost function exhibits weak cost complementarity. Part (b) of Theorem 5.3.1 shows that (N, f) is a convex game if the production cost function exhibits cost complementarity. Sharkey and Telser [1978] implicitly use the latter result in showing that cost complementarity implies that $c(x)$ is supportable, by basing a crucial construction on the greedy algorithm. (See Lemma 2.6.3.)

Theorem 5.3.1. *Consider the cooperative game (N, f) of the monopoly firm.*
(a) *If $x^{(i')} \wedge x^{(i'')} = 0$ for each pair of distinct consumers i' and i'' and if the production cost function $c(x)$ exhibits weak cost complementarity, then the cooperative game (N, f) is a convex game.*
(b) *If the production cost function $c(x)$ exhibits cost complementarity, then the cooperative game (N, f) is a convex game.*

Proof. Suppose that $x^{(i')} \wedge x^{(i'')} = 0$ for all distinct i' and i'' in N and that $c(x)$ exhibits weak cost complementarity. Pick any subsets S' and S'' of N. Note that $(\sum_{i\in S'} x^{(i)}) \vee (\sum_{i\in S''} x^{(i)}) = \sum_{i\in S'\cup S''} x^{(i)}$ and $(\sum_{i\in S'} x^{(i)}) \wedge (\sum_{i\in S''} x^{(i)}) = \sum_{i\in S'\cap S''} x^{(i)}$. Then

$$\begin{aligned} f(S'\cup S'') - f(S') &= -c\big(\textstyle\sum_{i\in S'} x^{(i)} + \sum_{i\in S''\setminus S'} x^{(i)}\big) + c\big(\sum_{i\in S'} x^{(i)}\big) \\ &\geq -c\big(\textstyle\sum_{i\in S'\cap S''} x^{(i)} + \sum_{i\in S''\setminus S'} x^{(i)}\big) + c\big(\sum_{i\in S'\cap S''} x^{(i)}\big) \\ &= f(S'') - f(S'\cap S''), \end{aligned}$$

where the inequality follows from weak cost complementarity (or, equivalently, the submodularity of $c(x)$). Therefore, (N, f) is a convex game, and part (a) holds.

The proof of part (b) is the same as that of part (a) except that the assumption that $x^{(i')} \wedge x^{(i'')} = 0$ for all distinct i' and i'' in N is dropped and the inequality follows from cost complementarity. □

5.3.2 Monotonic Surplus Sharing

A set $N = \{1, \ldots, n\}$ of agents (players) can operate either with the n agents acting independently or with the n agents acting cooperatively. If the agents act independently, then each agent i receives a return r_i. If the agents act cooperatively, then the n agents together receive a return q. Assume that $q > \sum_{i \in N} r_i$, so the n agents acting cooperatively receive a joint surplus $q - \sum_{i \in N} r_i > 0$ compared with what they would receive jointly if they act independently. When the agents act cooperatively, the **surplus sharing problem** involves determining a distribution y_i to each agent i such that the sum of the distributions to all the agents equals the return to the n agents acting cooperatively (that is, $\sum_{i \in N} y_i = q$) and each agent receives a distribution of at least what that agent would receive from acting independently (that is, $y_i \geq r_i$ for each i in N).

For each subset S of N, let $f(S) = \sum_{i \in S} r_i$ if $S \neq N$ and $f(N) = q$. The function $f(S)$ is supermodular if and only if $q \geq \sum_{i \in N} r_i$, so (N, f) is a convex game because it is here assumed that $q > \sum_{i \in N} r_i$. Furthermore, y is a solution for the surplus sharing problem if and only if y is in the core of the convex game (N, f). Therefore, an analysis of solutions for the surplus sharing problem can be based on consideration of the core of the convex game (N, f). For any permutation π of the agents N, the payoff vector $y(\pi)$ generated by the greedy algorithm for the convex game (N, f) has $y(\pi)_i = r_i$ for each i in $N \setminus \{\pi(n)\}$ and $y(\pi)_{\pi(n)} = q - \sum_{i \in N \setminus \{\pi(n)\}} r_i$. The payoff vectors $y(\pi)$ are identical for all $(n-1)!$ permutations π with the same agent as $\pi(n)$. The n distinct $y(\pi)$ are the extreme points of the core of (N, f) by part (a) of Theorem 5.2.4, and so they are the extreme points of the set of solutions for the surplus sharing problem. The Shapley value y' for the convex game (N, f) assigns a distribution $((n-1)/n)r_i + (1/n)(q - \sum_{k \in N \setminus \{i\}} r_k) = r_i + (1/n)(q - \sum_{k \in N} r_k)$ to each agent i, which gives to each agent i the amount r_i that the agent would receive acting independently plus an equal share of the surplus $q - \sum_{k \in N} r_k$.

Now suppose that the surplus sharing problem depends on a parameter t contained in a partially ordered set T, so that the returns r_i and q depend on the parameter t with this dependence denoted as r_i^t and q^t. The **monotonic surplus sharing problem** involves determining solutions y^t to the surplus sharing problem for each parameter t in T such that y^t is increasing in t. Moulin [1989] models and characterizes solutions to versions of the monotonic surplus sharing problem, but the hypotheses therein are rather different from those assumed here. For each parameter t, let $f^t(S)$ be the supermodular function on subsets S of N as defined in the preceding paragraph corresponding to the surplus sharing problem with parameter t. The set of solutions for the surplus sharing problem with parameter t is the core of the convex game (N, f^t). Assume that r_i^t and $q^t - \sum_{k \in N \setminus \{i\}} r_k^t$ are increasing in t for each agent i in N. (These are the conditions for t to be a complementary parameter for the collection of convex games (N, f^t).) Then each of the n distinct extreme points of the core of (N, f^t) (in particular, each extreme point given t generated by the greedy algorithm with a permutation π having $\pi(n) = i$ for given i in N) and the Shapley value of (N, f^t) are increasing in t and these are solutions for the monotonic surplus sharing problem. Indeed, any consistent method, as follows, for surplus sharing among the agents is a solution for the monotonic surplus sharing problem. Pick any $\alpha_i \geq 0$ for each i in N with $\sum_{i \in N} \alpha_i = 1$. Let the distribution to each agent i for the surplus sharing problem with parameter t be $r_i^t + \alpha_i(q^t - \sum_{k \in N} r_k^t) = (1 - \alpha_i) r_i^t + \alpha_i(q^t - \sum_{k \in N \setminus \{i\}} r_k^t)$, so each agent i receives the return that the agent would receive from acting independently plus a fraction α_i of the surplus. This is a solution for the monotonic surplus sharing problem, with the distinct extreme points of the core and the Shapley value being special cases. (See Theorem 5.2.7.)

5.3.3 Aircraft Landing Fee Game

Consider the problem of allocating runway construction costs to the flights using an airport. The present model, the observation that the cooperative game is a convex game, and the special form of the Shapley value given below are from Littlechild and Owen [1973] and Littlechild and Thompson [1977]. The set of flights (players) that might use the airport is $N = \{1, \ldots, n\}$. If flight i uses the airport, then a net operating profit (revenue minus operating cost) r_i is generated. Flight i requires a runway that would cost c_i to construct. The cost c_i increases with the length of the runway required for flight i. Length is the only salient property of the runway for flights that may use it. If i' and i'' are flights with $c_{i'} \leq c_{i''}$, then flight i'' requires a runway at least as long as the runway required for flight i' and flight i' could use a runway built to accommodate flight i''. The cost $g(S)$ of a runway built to accommodate a set of flights S is thus equal to

$$g(S) = \max_{i \in S} c_i,$$

the cost to build a runway to accommodate the flight in S with the longest runway requirement. By part (f) of Example 2.6.2, $g(S)$ is submodular on subsets S of all potential flights N. The net profit resulting from constructing and using a runway for a set of flights S is

$$\sum_{i \in S} r_i - \max_{i \in S} c_i.$$

Suppose that the airport administration wants to charge an aircraft landing fee z_i to the carrier for each flight i that uses the airport. The airport administration must choose these airport landing fees $z = (z_1, \ldots, z_n)$ to satisfy two kinds of constraints. The first constraint is that the total aircraft landing fees charged for building a runway to accommodate the set N of all flights equals the cost of building that runway; that is,

$$\textstyle\sum_{i \in N} z_i = g(N).$$

The second constraint is that for any subset S of flights it is not less expensive for the carriers operating the flights in S to build a separate airport to accommodate the flights S than it would be for the carriers to pay the landing fees for the flights S; that is,

$$\textstyle\sum_{i \in S} z_i \leq g(S) \text{ for each subset } S \text{ of } N.$$

A vector z of aircraft landing fees satisfies the airport administration's constraints if and only if $-z$ is in the core of the convex game $(N, -g)$.

By part (a) of Theorem 5.2.4, a vector z of aircraft landing fees corresponds to (the negative of) an extreme point of the core of $(N, -g)$ if and only if there is some permutation π of the flights N such that $z_{\pi(j)} = g(S(\pi, j)) - g(S(\pi, j-1)) = \max_{i \in S(\pi, j)} c_i - \max_{i \in S(\pi, j-1)} c_i$ for $j = 1, \ldots, n$. The aircraft landing fee for flight $\pi(j)$ is the additional cost to construct a runway to also accommodate the flight $\pi(j)$ given that the runway must accommodate the set of flights $S(\pi, j-1)$; that is, the difference between the cost of constructing a runway to accommodate the flights $S(\pi, j-1)$ and the cost of constructing a runway to accommodate the flights $S(\pi, j-1) \cup \{\pi(j)\} = S(\pi, j)$.

The aircraft landing fee corresponding to the Shapley value of the convex game $(N, -g)$ would charge for each flight i the average of $g(S(\pi, j)) - g(S(\pi, j-1))$ taken over all $n!$ permutations π of N and evaluated at that j for which $\pi(j) = i$. Assume that $c_i > 0$ for $i = 1, \ldots, n$ and define $\max_{i \in \emptyset} c_i = 0$. Let $h_1, \ldots, h_m$ be the distinct values among $\{c_1, \ldots, c_n\}$, and suppose that the elements of $h_1, \ldots, h_m$ are ordered so that $0 < h_1 < h_2 < \ldots < h_m$. Define $h_0 = 0$. Partition the set N of flights into disjoint subsets $N_1, \ldots, N_m$, where $N_k = \{i : i \in N, c_i = h_k\}$ for $k = 1, \ldots, m$. For a subset S of the flights N and $k = 1, \ldots, m$, define $g_k(S) = h_k - h_{k-1}$ if $S \cap (\cup_{p=k}^m N_p)$ is nonempty and $g_k(S) = 0$ if $S \cap (\cup_{p=k}^m N_p)$ is empty. Let $y'_{i,k}$ be the payoff to flight i according to the (negative of the) Shapley value in the convex game $(N, -g_k)$. Evidently, $y'_{i,k} = 0$ for each flight i in $\cup_{p=1}^{k-1} N_p$. The value of $y'_{i,k}$ is the same for each flight i in $\cup_{p=k}^{m} N_p$ by symmetry and $\sum_{i\in N} y'_{i,k} = g_k(N) = h_k - h_{k-1}$, so

$$y'_{i,k} = (h_k - h_{k-1})/|\cup_{p=k}^{m} N_p| \quad \text{for each } i \text{ in } \cup_{p=k}^{m} N_p.$$

Observe that $g(S) = \sum_{k=1}^{m} g_k(S)$ for each subset S of the flights N, and so the Shapley value for the convex game $(N, -g)$ is the sum of the Shapley values for the convex games $(N, -g_k)$ for $k = 1, \ldots, m$. Therefore, the aircraft landing fee to flight i in N corresponding to the Shapley value for the convex game $(N, -g)$ is

$$\sum_{k=1}^{k'} (h_k - h_{k-1})/|\cup_{p=k}^{m} N_p|$$

where k' is such that i is in $N_{k'}$ (that is, $h_{k'} = c_i$). These aircraft landing fees allocate the cost h_1 of the shortest possible runway requirement equally among all flights in N and for each $k = 2, \ldots, m$ the incremental cost $h_k - h_{k-1}$ of expanding the runway length from that required by flights in N_{k-1} to that required by flights in N_k is allocated equally among flights in $\cup_{p=k}^{m} N_p$ (which would require a runway at least as long as that costing h_k).

5.3.4 Trading Game

The trading game and its properties considered in this subsection are from Deng and Papadimitriou [1994]. Rosenthal [1988] considers a closely related convex game. Suppose that there is a set of countries (players) $N = \{1, \ldots, n\}$, and certain subsets of the countries can benefit from mutual trade. There are m nonempty subsets $Q_1, \ldots, Q_m$ of N such that the countries in each subset Q_k would receive a joint return $r_k \geq 0$ if the countries in Q_k could engage in mutual trade. Subsets of the countries N may form a trading block (coalition), so that any subsets of the countries in the trading block may engage in mutual trade but countries in the trading block may not trade with countries not in the trading block. The total return to a trading block S is

$$f(S) = \sum_{\{k: Q_k \subseteq S\}} r_k.$$

By inspection, as with part (i) of Example 2.6.2, $f(S)$ is supermodular in S and so (N, f) is a convex game.

For each permutation π of the countries N, the payoff

$$y(\pi)_{\pi(j)} = f(S(\pi, j)) - f(S(\pi, j-1)) = \sum_{\{k: \pi(j)\in Q_k, Q_k \subseteq S(\pi, j)\}} r_k$$

to country $\pi(j)$ is the sum of all the returns r_k to those subsets Q_k of countries for which the country $\pi(j)$ is the last country from Q_k to appear in the sequence $\pi(1), \ldots, \pi(n)$; that is those subsets Q_k that include $\pi(j)$ and are contained in $S(\pi, j)$.

If country i is in Q_k, then i is the last country from Q_k to appear in the sequence $\pi(1), \ldots, \pi(n)$ for a fraction $1/|Q_k|$ of the $n!$ permutations π of N and so the Shapley value y' takes the form

$$y'_i = \sum_{\{k:i \in Q_k\}} r_k/|Q_k|$$

for each country i.

The present model for trading games from Deng and Papadimitriou [1994] is a special case of activity selection games as developed by Topkis [1983] and considered in Subsection 5.5.4. (In an activity selection game, each coalition solves an optimization problem to select from among various activities available to it.) Deng and Papadimitriou [1994] study the problem, discussed in Subsection 5.2.3 for general convex games, of determining whether a particular payoff vector is in the core of a trading game. They construct a capacitated network such that the solution of a minimum cut problem in this network gives the answer to the membership problem in the trading game. That construction is a special case of a construction given by Topkis [1983] and presented in Subsection 5.5.4 for determining membership in an activity selection game through the solution of a corresponding minimum cut problem. (See Theorem 5.5.1.)

5.4 Activity Optimization Games

This section formulates and analyzes a general class of cooperative games, activity optimization games with complementarity. This is a class of cooperative games for which the model captures optimal decision making by each coalition of players. Each such cooperative game is a convex game, and a variety of monotone comparative statics results hold. The model and results of this section are based on Topkis [1987], except for the generalization specified in the following paragraph.

There is a set $N = \{1, \ldots, n\}$ of players. For each player i in N, there are $m_i \geq 0$ private activities for which player i chooses activity levels. Let $x_{(i)}$ be the m_i-vector of activity levels for the private activities of player i. There are $m_0 \geq 0$ public activities such that any coalition can operate these activities and choose their activity levels. Let $x_{(0)}$ be the m_0-vector of activity levels for the public activities. The private activities of players not in a coalition are unavailable to that coalition and cannot operate at a positive level for the coalition. A coalition decides the levels of its members' private activities and the levels of the public activities by choosing these levels to maximize its return function. Let $m = \sum_{i=0}^{n} m_i$. The m-vector $x = (x_{(0)}, x_{(1)}, \ldots, x_{(n)})$ gives the levels of all private and public activities. The m-vector of activity levels that any coalition chooses must be contained in a specified set L of feasible activity levels, where L is a subset of R^m, the 0 vector is in L, and 0 is a lower bound for L. (Recall that 0 denotes a vector of 0's with the dimension being clear from context.) If coalition S chooses a vector x of activity levels, then $x_{(i)} = 0$ for each player i not in S. Let $X(S)$ be the set of all vectors of activity levels that are feasible for coalition S, so $X(S) = \{x : x \in L, x_{(i)} = 0 \text{ for each } i \text{ in } N \setminus S\}$. Because the 0 vector of activity levels is feasible for any coalition, each $X(S)$ is nonempty. A coalition S that chooses a feasible vector x of activity levels from $X(S)$ receives a return $g(x, S)$, where $g(x, S)$ is a real-valued function that is bounded above for subsets S of N and x in $X(S)$. (This extends the model from that considered in Topkis [1987], which requires $g(x, S)$ to be separable in x and S as $g(x, S) = g_1(x) + g_2(S)$.) Assume that a net return of 0 results from all activities operating at 0 level with no players in a coalition and that a positive net return cannot be realized by using only the public activities, so $g(0, \emptyset) = 0$ and $\sup_{x \in X(\emptyset)} g(x, \emptyset) = 0$. Each coalition seeks to maximize its return by choosing an optimal feasible vector of activity levels. The supremum of all net returns that a coalition S could realize is the characteristic function $f(S)$. Thus for each subset S of N,

$$f(S) = \sup_{x \in X(S)} g(x, S). \qquad (5.4.1)$$

The above assumptions imply that each $f(S)$ is finite and $f(\emptyset) = 0$. The cooperative game (N, f) is an **activity optimization game**. Denote an activity optimization game by the triple (N, g, L). An activity optimization game (N, g, L) is **playable** if L is compact and $g(x, S)$ is upper semicontinuous in x on $X(S)$ for each subset S of N. For a playable activity selection game, the supremum in (5.4.1) is attained and there exists an optimal vector of activity levels for each coalition.

An **activity optimization game with complementarity** is an activity optimization game (N, g, L) for which L is a sublattice of R^m and $g(x, S)$ is supermodular in (x, S) on the sublattice $\{(x, S) : S \subseteq N, x \in X(S)\}$ of $L \times \mathcal{P}(N)$. For an activity optimization game with complementarity, all activities are complementary, the players are complementary in any interactions other than those resulting from their choices of activity levels, and each activity that is not a private activity of a particular player is complementary with the presence of that player in a coalition. Also, $X(S)$ is increasing in S for subsets S of N by part (b) of Example 2.4.1.

Theorem 5.4.1 shows that an activity optimization game with complementarity is a convex game; that is, the complementarities noted in the preceding paragraph lead to complementarity among the players.

Theorem 5.4.1. *An activity optimization game with complementarity is a convex game.*

Proof. Let (N, g, L) be an activity optimization game with complementarity. Then $g(x, S)$ is supermodular in (x, S) on the sublattice $\{(x, S) : S \subseteq N, x \in X(S)\}$, and $f(S)$ is supermodular in S by (5.4.1) and Theorem 2.7.6. □

Theorem 5.4.2 demonstrates that optimal activity levels increase as a function of coalition membership in an activity optimization game with complementarity.

Theorem 5.4.2. *If (N, g, L) is an activity optimization game with complementarity, then the set* $\text{argmax}_{x \in X(S)} g(x, S)$ *of optimal activity levels for a given subset S of the players N is increasing in S. If, in addition, the activity optimization game is playable, then there exists a greatest (least) vector of optimal activity levels for each subset S of the players and this greatest (least) vector of optimal activity levels is increasing in S.*

Proof. The first statement of this result follows from Theorem 2.8.2. The second statement follows from part (a) of Theorem 2.8.3. □

Suppose that (N, g, L) is a playable activity optimization game with complementarity, so the corresponding cooperative game (N, f) is a convex game by Theorem 5.4.1. In order to implement the greedy algorithm with a permutation π of the players N, it is necessary to compute $f(S(\pi, 1)), f(S(\pi, 2)), \ldots, f(S(\pi, n))$. This involves solving the optimization problem (5.4.1) n times, once each for the coalition S being $S(\pi, 1)$, $S(\pi, 2)$, $\ldots$, $S(\pi, n)$. Suppose that one computes $f(S(\pi, 1))$, $f(S(\pi, 2))$, $\ldots$, $f(S(\pi, n))$ in this same order. The result of Theorem 5.4.2 that optimal activity levels are an increasing function of the membership of the coalition indicates how one may reduce the domains of some of these n optimization problems. When it is time to compute $f(S(\pi, j))$ for some j with $1 < j \leq n$, one has already computed $f(S(\pi, j-1))$ and that computation ought to provide some x' in $\text{argmax}_{x \in S(\pi, j-1)} g(x, S(\pi, j-1))$. Theorem 5.4.2 implies that $\text{argmax}_{x \in S(\pi, j)} g(x, S(\pi, j)) \cap [x', \infty)$ is nonempty. Thus one can compute $f(S(\pi, j))$ by maximizing $g(x, S(\pi, j))$ over $X(S(\pi, j)) \cap [x', \infty)$ instead of over all of $X(S(\pi, j))$, with a potential for decreasing the required computational effort due to a smaller feasible region for the maximization.

Suppose that T is a partially ordered set and (N, g^t, L) is an activity optimization game for each t in T with the function $g^t(x, S)$ depending on the parameter t. Then t is a **complementary parameter** if $g^t(x, S)$ has increasing differences in $((x, S), t)$ on $\{(x, S) : S \subseteq N, x \in X(S)\} \times T$; that is, if the activity levels x and the parameter t are complements in the sense that (with respect to the return $g^t(x, S)$) higher activity levels are relatively more desirable for a higher parameter and if the coalition and the parameter are complements in the sense that (with respect to the return $g^t(x, S)$) a larger coalition is relatively more desirable for a higher parameter. The characteristic function of the activity optimization game (N, g^t, L) is denoted $f^t(S)$ for each parameter t in T.

Theorem 5.4.3 shows that optimal activity levels increase with the parameter for a collection of activity optimization games with complementarity having a complementary parameter.

Theorem 5.4.3. *If T is a partially ordered set, (N, g^t, L) is an activity optimization game with complementarity for each t in T, and t is a complementary parameter, then the set* $\operatorname{argmax}_{x\in X(S)} g^t(x, S)$ *of optimal activity levels for a given subset S of the players and for a given parameter t is increasing in t. If, in addition, the activity optimization game is playable, then there exists a greatest (least) vector of optimal activity levels for each subset S of the players and each parameter t and this greatest (least) vector of optimal activity levels is increasing in t.*

Proof. The first statement of this result follows from Theorem 2.8.1. The second statement follows from part (a) of Theorem 2.8.3. □

For a collection of activity optimization games with complementarity having a complementary parameter, Theorem 5.4.4 shows that the parameter is a complementary parameter for the corresponding collection of cooperative games in characteristic function form.

Theorem 5.4.4. *If T is a partially ordered set, (N, g^t, L) is an activity optimization game with complementarity for each t in T, and t is a complementary parameter, then t is a complementary parameter for the cooperative game (N, f^t).*

Proof. Pick any t' and t'' in T with $t' \preceq t''$. By the present hypotheses and Theorem 2.6.2, $g^t(x, S)$ is supermodular in (x, S, t) on the lattice $\{(x, S, t) : S \subseteq N, x \in X(S), t \in \{t', t''\}\}$. By Theorem 2.7.6, $f^t(S) = \sup_{x\in X(S)} g^t(x, S)$ is supermodular in (S, t) on $\mathcal{P}(N) \times \{t', t''\}$. By Theorem 2.6.1, $f^t(S)$ has increasing differences in (S, t) on $\mathcal{P}(N) \times \{t', t''\}$. Therefore, t is a complementary parameter for the cooperative game (N, f^t). □

Theorem 5.4.4 and Theorem 5.2.7 imply the following.

Corollary 5.4.1. *If T is a partially ordered set, (N, g^t, L) is an activity optimization game with complementarity for each t in T, and t is a complementary parameter, then the payoff vector generated by the greedy algorithm for each permutation of the players is an increasing function of t on T and the Shapley value is an increasing function of t on T.*

If (N, g^t, L) is an activity optimization game for each t in a lattice T, and $g^t(X, S)$ is supermodular in t on T for each subset S of N and each x in $X(S)$, then t is a **supermodular parameter**. Theorem 5.4.5 gives conditions for a parameterized collection of activity optimization games to have a characteristic function that is a supermodular function of the parameter.

Theorem 5.4.5. *Suppose that (N, g^t, L) is an activity optimization game with complementarity for each t in a set T and that t is a complementary parameter and supermodular parameter. Then $f^t(S)$ is supermodular in t on T for each subset S of N.*

Proof. Pick any subset S of N. By Theorem 2.6.2, $g^t(x, S)$ is supermodular in (x, t) on $X(S) \times T$. By (5.4.1) and Theorem 2.7.6, $f^t(S) = \sup_{x \in X(S)} g^t(x, S)$ is supermodular in t on T. □

5.5 Examples of Activity Optimization Games

This section gives examples of activity optimization games with complementarity, so the results of Section 5.4 apply for each of these examples. Furthermore, each of these examples of activity optimization games with complementarity is a convex game by Theorem 5.4.1, and so the results of Section 5.2 apply to each example. The significant distinction between these examples and the examples of convex games given in Section 5.3 is that these examples incorporate modeling of optimal decision making by each coalition of players. Subsection 5.5.1 considers a general welfare game in which players consume public goods and private goods and determine the consumption levels of the public goods and the private goods. Subsection 5.5.2 considers a collection of firms that may engage in common procurement of inputs for their respective production processes. Subsection 5.5.3 considers firms that may jointly invest in shared production facilities for their respective production processes. Subsection 5.5.4 considers an activity selection game, generalizing the selection problem of Subsection 3.7.3 to the context of a cooperative game. Subsection 5.5.5 considers a waste disposal game, where firms involved in production on the shore of a common lake must determine the level of polluting waste that they dump in the water.

5.5.1 General Welfare Game

The following general welfare game extends a model studied by Sharkey [1982b] to additionally include public goods and so incorporate features of a cooperative game model of Moulin [1990] concerned with the cost-sharing of public goods. There is a set $N = \{1, \ldots, n\}$ of consumers (players), with each consumer consuming k_0 public goods and k private goods. The k_0-vector $x_{(0)}$ of public goods and the k-vector $x_{(i)}$ of private goods consumed by consumer i must be contained, respectively, in subsets L_0 of R^{k_0} and L_i of R^k. The 0 vector is in L_0 and in L_i, $x_{(0)} \geq 0$ for each $x_{(0)}$ in L_0, and $x_{(i)} \geq 0$ for each $x_{(i)}$ in L_i. Any coalition that forms determines the levels of the public goods and produces jointly the consumption vectors of private goods for its members. The cost of a k_0-vector $z \geq 0$ of public goods available to each member of a coalition is $c(z)$, where $c(z) \geq 0 = c(0)$ for each z. The cost of producing a k-vector $w \geq 0$ of private goods is $b(w)$, where $b(0) = 0$. The utility to consumer i of a k_0-vector $x_{(0)}$ of public goods and a k-vector $x_{(i)}$ of private goods is $r_i(x_{(0)}, x_{(i)})$, where $r_i(x_{(0)}, 0) = 0$ for each $x_{(0)}$. Let $x = (x_{(0)}, x_{(1)}, \ldots, x_{(n)})$ be the vector composed of the vector of public goods and the consumption vectors of the n consumers, let $L = \times_{i \in N \cup \{0\}} L_i$ be the set of all feasible vectors of public goods and individual consumption for the set N of all consumers, and let $X(S) = \{x : x \in L, x_{(i)} = 0 \text{ for each } i \in N \setminus S\}$ be the set of all feasible vectors composed of a vector of public goods and individual consumption for each subset S of N. For x in L and any subset S of N, let

$$g(x, S) = \textstyle\sum_{i \in N} r_i(x_{(0)}, x_{(i)}) - \sum_{i \in N \setminus S} r_i(0, 0) - c(x_{(0)}) - b(\sum_{i \in N} x_{(i)}).$$

Assume that $g(x, S)$ is bounded above on $L \times \mathcal{P}(N)$. For each subset S of N, the characteristic function is

$$\begin{aligned} f(S) &= \sup\big\{\textstyle\sum_{i \in S} r_i(x_{(0)}, x_{(i)}) - c(x_{(0)}) - b\big(\sum_{i \in S} x_{(i)}\big) : \\ &\qquad x_{(i)} \in L_i \text{ for each } i \in S \cup \{0\}\big\} \\ &= \sup\big\{\textstyle\sum_{i \in N} r_i(x_{(0)}, x_{(i)}) - \sum_{i \in N \setminus S} r_i(0, 0) - c(x_{(0)}) - b(\sum_{i \in N} x_{(i)}) : \\ &\qquad x \in L, x_{(i)} = 0 \text{ for each } i \in N \setminus S\big\} \\ &= \sup_{x \in X(S)} g(x, S). \end{aligned}$$

Then (N, L, g) is an activity optimization game. Suppose also that each L_i is compact, $c(z)$ and $b(w)$ are lower semicontinuous, and each $r_i(x_{(0)}, x_{(i)})$ is upper semicontinuous, so the activity optimization game is playable.

Suppose that, in addition, L_0 is a sublattice of R^{k_0}, L_i is a sublattice of R^k for each i in N, $c(z)$ is submodular (equivalently, $c(z)$ exhibits weak cost complementarity), $b(w)$ is concave in each individual component and is submodular (equivalently, $b(w)$ exhibits cost complementarity), and each $r_i(x_{(0)}, x_{(i)})$ is supermodular. Then this is an activity optimization game with complementarity. By part (b) of Example 2.2.7, the restriction of $x_{(i)}$ to a sublattice of R^k for i in N permits bounds on the difference between the consumption of any private good and a nonnegative scalar times the consumption of any other private good. Likewise for the restriction of the public goods vector $x_{(0)}$ to a sublattice of R^{k_0}. The marginal cost of producing each private good is a decreasing function of the existing levels of production of each private good. There are economies of scope for public and private goods as well as increasing returns to scale in production of private goods. The supermodularity of $r_i(x_{(0)}, x_{(i)})$ implies by Theorem 2.6.1 that the public and private goods are complementary in the sense that the marginal utility to any consumer of any one good is an increasing function of the consumption level of each of the other goods. By Theorem 5.4.1, this is a convex game as Sharkey [1982b] shows for the case with no public goods ($k_0 = 0$). By Theorem 5.4.2, there exist optimal levels of public goods and private goods for each coalition S that are an increasing function of S.

Now suppose that, in addition, each public goods cost function $c(z)$, each private goods production cost function $b(w)$, and each utility function $r_i(x_{(0)}, x_{(i)})$ depend on a parameter t contained in a partially ordered set T, and denote this dependence by $c^t(z)$, $b^t(w)$, and $r_i^t(x_{(0)}, x_{(i)})$. Assume that $c^t(z)$ has decreasing differences in (z, t), $b^t(w)$ has decreasing differences in (w, t), $r_i^t(x_{(0)}, x_{(i)})$ has increasing differences in $((x_{(0)}, x_{(i)}), t)$ for each i, and $r_i^t(0, 0)$ is increasing in t for each i, so t is a complementary parameter. Theorem 5.4.3 implies that optimal levels of the public goods and the private goods for each coalition are an increasing function of the parameter. Corollary 5.4.1 (together with part (a) of Theorem 5.2.4) implies that the extreme points of the core and the Shapley value are increasing functions of the parameter. Assume also that T is a lattice, $c^t(z)$ is submodular in t for each z, $b^t(w)$ is submodular in t for each w, $r_i^t(x_{(0)}, x_{(i)})$ is supermodular in t for all i and $(x_{(0)}, x_{(i)})$, and $r_i^t(0, 0)$ does not depend on t for each i, so the parameter t is a supermodular parameter. By Theorem 5.4.5, the characteristic function for each coalition is a supermodular function of the parameter.

5.5.2 Production Game with Common Procurement of Inputs

Consider a set $N = \{1, \ldots, n\}$ of firms (players), each of which produces a single product. The n production processes of the various firms use k inputs denoted $j = 1, \ldots, k$. The production of each unit of the product of firm i requires $a_{ij} \geq 0$ units of input j. The production level x_i of firm i must be contained in a subset L_i of $[0, \infty)$, where 0 is in L_i. The production of x_i by firm i results in a return $r_i(x_i)$, before taking into account the cost of inputs. If a coalition forms, its members make joint purchases of inputs. The cost of z units of input j is $c_j(z)$, where $c_j(0) = 0$. Let $x = (x_1, \ldots, x_n)$ be the vector of production levels for the n firms, let $L = \times_{i \in N} L_i$ be the set of feasible production levels for the n firms N, and let $X(S) = \{x : x \in L,$ $x_i = 0$ for each $i \in N \setminus S\}$ be the set of feasible production levels for a subset S of the firms N. For a feasible joint production vector x in L and any subset S of the firms, let

$$g(x, S) = \textstyle\sum_{i \in N} r_i(x_i) - \sum_{i \in N \setminus S} r_i(0) - \sum_{j=1}^{k} c_j(\sum_{i \in N} a_{ij} x_i).$$

Assume that $g(x, S)$ is bounded above on $L \times \mathcal{P}(N)$. Then the characteristic function for any subset S of N is

$$\begin{aligned} f(S) &= \sup\left\{\textstyle\sum_{i \in S} r_i(x_i) - \sum_{j=1}^{k} c_j\left(\sum_{i \in S} a_{ij} x_i\right) : x_i \in L_i \text{ for each } i \in S\right\} \\ &= \sup\left\{\textstyle\sum_{i \in N} r_i(x_i) - \sum_{i \in N \setminus S} r_i(0) - \sum_{j=1}^{k} c_j\left(\sum_{i \in N} a_{ij} x_i\right) : \right. \\ &\qquad \left. x \in L, x_i = 0 \text{ for each } i \in N \setminus S\right\} \\ &= \sup\nolimits_{x \in X(S)} g(x, S). \end{aligned}$$

Then (N, L, g) is an activity optimization game. Suppose also that each L_i is compact, each $r_i(x_i)$ is upper semicontinuous, and each $c_j(z)$ is lower semicontinuous, so the activity optimization game is playable.

Suppose that, in addition, each input procurement cost function $c_j(z)$ is concave. A coalition undertaking common procurement can then take advantage of the decreasing marginal cost of each input. The resulting cost savings due to common procurement encourage coalition formation. This is an activity optimization game with complementarity by Theorem 2.6.4, part (a) of Lemma 2.6.2, and part (b) of Lemma 2.6.1. By Theorem 5.4.1, this is a convex game. By Theorem 5.4.2, optimal production levels for the firms in any coalition S are an increasing function of S.

Now suppose that, in addition, each return function $r_i(x_i)$ and each input procurement cost function $c_j(z)$ depends on a parameter t contained in a partially ordered set T, and denote this dependence by $r_i^t(x_i)$ and $c_j^t(z)$. Assume that $r_i^t(x_i)$ has increasing differences in (x_i, t) for each i, $c_j^t(z)$ has decreasing differences in (z, t) for each j, and $r_i^t(0)$ is increasing in t for each i, so t is a complementary parameter. Theorem 5.4.3 implies that optimal production levels for each coalition are an increasing function of the parameter. Corollary 5.4.1 (together with part (a) of Theorem 5.2.4) implies that the extreme points of the core and the Shapley value are increasing functions of the parameter. Assume also that T is a lattice, $r_i^t(x_i)$ is supermodular in t for all i and x_i, $c_j^t(z)$ is submodular in t for all j and z, and $r_i^t(0)$ does not depend on t for each i, so the parameter t is a supermodular parameter. By Theorem 5.4.5, the characteristic function for each coalition is a supermodular function of the parameter. The conditions for a supermodular parameter would hold if T is a sublattice of R^{n+k}, each return function $r_i^t(x_i)$ depends on t only through the component t_i of t, and each input procurement cost function $c_j^t(z)$ depends on t only through the component t_{n+j}.

5.5.3 Investment and Production Game

Consider a set $N = \{1, \ldots, n\}$ of firms (players), each of which produces an identical product. The production level x_i of each firm i must be contained in a subset L_i of $[0, \infty)$, where 0 is in L_i. Any coalition that forms chooses an amount x_0 to invest in shared production facilities. The investment x_0 must be contained in a subset L_0 of $[0, \infty)$, where 0 is in L_0. The shared production facilities are available for use in the production processes of each firm in the coalition. The production of x_i by firm i results in a return $r_i(x_i)$ before taking into account the amount invested in shared production facilities or the cost of production. If firm i is a member of a coalition that invests an amount x_0 in shared production facilities and if firm i produces x_i, then there is a nonnegative production cost $c_i(x_i, x_0)$ for firm i, where $c_i(0, x_0) = 0$ for each x_0 in L_0. Let $x = (x_0, x_1, \ldots, x_n)$ be the vector composed of the amount invested in shared production facilities and the production levels of all firms in N, let $L = \times_{i \in N \cup \{0\}} L_i$ be the set of feasible values for x given a coalition consisting of all firms N, and let $X(S) = \{x : x \in L, x_i = 0 \text{ for each } i \in N \setminus S\}$ be the set of feasible values for x given a coalition consisting of any subset S of the firms. For x in L and any subset S of N, let

$$g(x, S) = \textstyle\sum_{i \in N} r_i(x_i) - \sum_{i \in N \setminus S} r_i(0) - \sum_{i \in N} c_i(x_i, x_0) - x_0.$$

Assume that $g(x, S)$ is bounded above on $L \times \mathcal{P}(N)$. For any subset S of N, the characteristic function is

$$\begin{aligned} f(S) &= \sup\left\{\textstyle\sum_{i \in S} r_i(x_i) - \sum_{i \in S} c_i(x_i, x_0) - x_0 : x_i \in L_i \text{ for each } i \in S \cup \{0\}\right\} \\ &= \sup\left\{\textstyle\sum_{i \in N} r_i(x_i) - \sum_{i \in N \setminus S} r_i(0) - \sum_{i \in N} c_i(x_i, x_0) - x_0 :\right. \\ &\qquad \left. x \in L, x_i = 0 \text{ for each } i \in N \setminus S\right\} \\ &= \sup\nolimits_{x \in X(S)} g(x, S). \end{aligned}$$

Then (N, L, g) is an activity optimization game. Suppose also that each L_i is compact, each $r_i(x_i)$ is upper semicontinuous, and each $c_i(x_i, x_0)$ is lower semicontinuous, so the activity optimization game is playable.

Suppose that, in addition, $c_i(x_i, x_0)$ is submodular in (x_i, x_0) on $L_i \times L_0$ for each firm i, so this is an activity optimization game with complementarity. The marginal cost of production for each firm decreases with the investment in shared production facilities by Theorem 2.6.1. By Theorem 5.4.1, this is a convex game. By Theorem 5.4.2, optimal investment and production levels for each coalition S are an increasing function of S.

Now suppose that, in addition, each return function $r_i(x_i)$ and each production cost function $c_i(x_i, x_0)$ depends on a parameter t contained in a partially ordered set T, and denote this dependence by $r_i^t(x_i)$ and $c_i^t(x_i, x_0)$. Assume that $r_i^t(x_i)$ has increasing differences in (x_i, t) for each i, $c_i^t(x_i, x_0)$ has decreasing differences in $((x_i, x_0), t)$ for each i, and $r_i^t(0)$ is increasing in t for each i, so t is a complementary parameter. Theorem 5.4.3 implies that optimal investment and production levels for each coalition are an increasing function of the parameter. Corollary 5.4.1 (together with part (a) of Theorem 5.2.4) implies that the extreme points of the core and the Shapley value are increasing functions of the parameter. Assume also that T is a lattice, $r_i^t(x_i)$ is supermodular in t for all i and x_i, $c_I^t(x_i, x_0)$ is submodular in t for all i and (x_i, x_0), and $r_i^t(0)$ does not depend on t for each i, so t is a supermodular parameter. By Theorem 5.4.5, the characteristic function for each coalition is a supermodular function of the parameter.

5.5.4 Activity Selection Game

This subsection considers a version of the selection problem, extended to the context of a cooperative game. (See Subsection 3.7.3.) The results stated in this subsection are developed in Topkis [1983], which emphasizes the use of associated minimum cut problems to carry out optimization in the cooperative game and to analyze qualitative properties of the cooperative game and its solutions. The present analysis based on properties of an activity optimization game with complementarity is from Topkis [1987]. (The trading game model of Deng and Papadimitriou [1994] considered in Subsection 5.3.4 is a special case of an activity selection game.)

Consider a set $N = \{1, \dots, n\}$ of firms (players), where each firm i has available a finite set A_i of private activities from which to select. There is a finite set A_0 of public activities that are available to any coalition. The sets A_i for i in $N \cup \{0\}$ are disjoint. A coalition can select activities that are private activities of its members or that are public activities, but it cannot select a private activity of a firm not in the coalition. The decision problem for a coalition S involves selecting from $\cup_{i\in S\cup\{0\}} A_i$ a subset of activities in which to engage. Let $m_i = |A_i|$ for i in $N \cup \{0\}$ and $m = \sum_{i=0}^{n} m_i$. Let L be the set of all m-vectors of 0's and 1's, with each component of L corresponding to a distinct element of $\cup_{i\in N\cup\{0\}} A_i$. There is a one-to-one relationship between vectors x in L and selections of activities, where $x_j = 1$ if and only if activity j is selected. Each of the m activities can be viewed as operating either at level 0 or at level 1, where an activity operates at level 0 if it is not selected or at level 1 if it is selected. The set of feasible activity levels for a coalition S is $X(S) = \{x : x \in L, x_j = 0 \text{ for each } j \in A_i \text{ with } i \in N \setminus S\}$. A coalition that selects any activity j receives a return r_j, where r_j may be positive, negative, or zero. For any distinct activities j' and j'', a coalition incurs a nonnegative cost $b_{j',j''}$ if it selects j' but not j''. These costs can induce a coalition to select activities with negative returns. For a vector of activity levels x in L, the resulting profit is

$$g(x) = \textstyle\sum_{j\in\cup_{i\in N\cup\{0\}} A_i} r_j x_j - \sum_{j'\in\cup_{i\in N\cup\{0\}} A_i} \sum_{j''\in\cup_{i\in N\cup\{0\}} A_i} b_{j',j''} x_{j'}(1 - x_{j''}).$$

(If a particular coalition S forms, the profit $g(x)$ from the vector of activity levels x does not depend on S. However, the set $X(S)$ of feasible activity levels for S does depend on S.) Assume that $g(x) \le 0$ for each x in $X(\emptyset)$, so the empty coalition (having available only the public activities in A_0) cannot earn a positive profit. By Corollary 2.6.1, $g(x)$ is supermodular on L, and the various activities are complementary. For any subset S of N, the characteristic function is

$$\begin{aligned} f(S) &= \max\{\textstyle\sum_{j\in\cup_{i\in N\cup\{0\}} A_i} r_j x_j - \sum_{j'\in\cup_{i\in N\cup\{0\}} A_i} \sum_{j''\in\cup_{i\in N\cup\{0\}} A_i} b_{j',j''} x_{j'}(1 - x_{j''}) : \\ &\qquad x \in L, x_j = 0 \text{ for each } j \in \cup_{i\in N\setminus S} A_i\} \\ &= \max_{x\in X(S)} g(x). \end{aligned}$$

Then (N, L, g) is a playable activity optimization game with complementarity. Each coalition solves a selection problem, and the optimal selection of activities and the value of the characteristic function for each coalition can be found by solving an associated minimum cut problem as in Lemma 3.7.5. Therefore, applying the greedy algorithm involves solving n minimum cut problems.

By Theorem 5.4.1, this is a convex game. By Theorem 5.4.2, optimal selections of public activities and private activities for each coalition S are an increasing function of S.

Now suppose that each return r_j depends on a parameter t contained in a partially ordered set T, and denote this dependence by r_j^t. Assume that r_j^t is increasing in t for each j, so t is a complementary parameter. Theorem 5.4.3 implies that optimal selections of public activities and private activities for each coalition are an increasing function of the parameter. (See also Theorem 3.7.6.) Corollary 5.4.1 (together with part (a) of Theorem 5.2.4) implies that the extreme points of the core and the Shapley value are increasing functions of the parameter. Assume also that T is a lattice and r_j^t is supermodular in t for each j, so t is a supermodular parameter. By Theorem 5.4.5, the characteristic function for each coalition is a supermodular function of the parameter. One case of t being a supermodular parameter would be if T is a sublattice of R^m and each r_j^t depends only on t_j. (See also Theorem 3.7.5.) In particular, if t is the m-vector $\{r_j : j \in \cup_{i \in N \cup \{0\}} A_i\}$ of returns for all m activities, then t is a complementary parameter and a supermodular parameter and the preceding conclusions based on Theorem 5.4.3, Corollary 5.4.1, and Theorem 5.4.5 hold.

Now suppose that an activity selection game is given. Consider the membership problem, discussed in Subsection 5.2.3 for general convex games, to determine whether a particular payoff vector is in the core. A capacitated network is constructed such that the solution of a minimum cut problem determines the solution for the membership problem. If the payoff vector is not in the core, then a minimum cut indicates an inequality describing the core such that the payoff vector fails to satisfy that inequality. Theorem 5.5.1 confirms these properties. This appears to give a significant computational improvement over applying the ellipsoid method of Grötschel, Lovász, and Schrijver [1981] to this particular convex game.

Let y' be a feasible payoff vector, so $\sum_{i \in N} y'_i = f(N)$. Determining whether y' is in the core is equivalent to determining whether y' is acceptable; that is, whether $\sum_{i \in S} y'_i \geq f(S)$ for each subset S of the firms N. Construct a network with $n + m + 2$ nodes consisting of a source node s, a sink node t, a node i corresponding to each firm i in N, a node j corresponding to each activity j in $\cup_{i \in N \cup \{0\}} A_i$, and a directed edge joining each ordered pair of nodes. For each i in N with $y'_i < 0$, there is a capacity $-y'_i$ on the edge from node s to node i. For each i in N with $y'_i > 0$, there is a capacity y'_i on the edge from node i to node t. For each j in $\cup_{i \in N \cup \{0\}} A_i$ with $r_j > 0$, there is a capacity r_j on the edge from node s to node j. For each j in $\cup_{i \in N \cup \{0\}} A_i$ with $r_j < 0$, there is a capacity $-r_j$ on the edge from node j to node t. For all distinct j' and j'' in $\cup_{i \in N \cup \{0\}} A_i$, there is a capacity $b_{j',j''}$ on the edge from node j' to node j''. For all i in N and j in A_i, there is a capacity M on the edge from node j to node i where $M > \sum_{\{j : j \in \cup_{i \in N \cup \{0\}} A_i, r_j > 0\}} r_j$. Each other edge has capacity 0. For a payoff vector y, a subset S of N, and a subset X of $\cup_{i \in N \cup \{0\}} A_i$, let

$$h(y, S, X) = -\sum_{i \in S} y_i + \sum_{j \in X} r_j - \sum_{j' \in X} \sum_{j'' \in (\cup_{i \in N \cup \{0\}} A_i) \setminus X} b_{j',j''} - |X \cap (\cup_{i \in N \setminus S} A_i)| M.$$

Theorem 5.5.1. *Suppose that $S' \cup X' \cup \{s\}$ is a minimum cut in the above network with feasible payoff vector y', where S' is a subset of N and X' is a subset of $\cup_{i \in N \cup \{0\}} A_i$. If $h(y', S', X') = 0$, then the payoff vector y' is in the core. If $h(y', S', X') > 0$, then $\sum_{i \in S'} y'_i < f(S')$ and the payoff vector y' is not in the core.*

Proof. For any subset S of N,

$$\begin{aligned} f(S) &= \max_{X \subseteq \cup_{i \in S \cup \{0\}} A_i} \big(\textstyle\sum_{j \in X} r_j - \sum_{j' \in X} \sum_{j'' \in (\cup_{i \in N \cup \{0\}} A_i) \setminus X} b_{j',j''}\big) \\ &= \max_{X \subseteq \cup_{i \in N \cup \{0\}} A_i} \big(\textstyle\sum_{j \in X} r_j - \sum_{j' \in X} \sum_{j'' \in (\cup_{i \in N \cup \{0\}} A_i) \setminus X} b_{j',j''} \\ &\qquad - |X \cap (\cup_{i \in N \setminus S} A_i)| M\big) \\ &= \textstyle\sum_{i \in S} y'_i + \max_{X \subseteq \cup_{i \in N \cup \{0\}} A_i} h(y', S, X) \end{aligned}$$

and so $\sum_{i \in S} y'_i \geq f(S)$ if and only if $0 \geq h(y', S, X)$ for each subset X of $\cup_{i \in N \cup \{0\}} A_i$. Hence, y' is in the core if and only if $0 \geq h(y', S, X)$ for each subset S of N and each subset X of $\cup_{i \in N \cup \{0\}} A_i$.

For any subset S of N and any subset X of $\cup_{i \in N \cup \{0\}} A_i$, the capacity of the cut $S \cup X \cup \{s\}$ in the above network is

$$\begin{aligned} &- \textstyle\sum_{\{i: i \in N \setminus S, y'_i < 0\}} y'_i + \sum_{\{i: i \in S, y'_i > 0\}} y'_i + \sum_{\{j: j \in (\cup_{i \in N \cup \{0\}} A_i) \setminus X, r_j > 0\}} r_j - \sum_{\{j: j \in X, r_j < 0\}} r_j \\ &\quad + \textstyle\sum_{j' \in X} \sum_{j'' \in (\cup_{i \in N \cup \{0\}} A_i) \setminus X} b_{j',j''} + |X \cap (\cup_{i \in N \setminus S} A_i)| M \\ &= - \textstyle\sum_{\{i: i \in N, y'_i < 0\}} y'_i + \sum_{\{j: j \in \cup_{i \in N \cup \{0\}} A_i, r_j > 0\}} r_j - h(y', S, X). \end{aligned}$$

Therefore, $S' \cup X' \cup \{s\}$ is a minimum cut in the above network if and only if (S', X') maximizes $h(y', S, X)$ over all subsets S of N and all subsets X of $\cup_{i \in N \cup \{0\}} A_i$, and y' is in the core if and only if $0 \geq h(y', S', X')$ where $S' \cup X' \cup \{s\}$ is a minimum cut. Because $h(y', \emptyset, \emptyset) = 0$, $h(y', S', X') = \max_{S \subseteq N, X \subseteq \cup_{i \in N \cup \{0\}} A_i} h(y', S, X) \geq 0$. Then y' is in the core if $h(y', S', X') = 0$, and y' is not in the core if $h(y', S', X') > 0$. Furthermore, if $h(y', S', X') > 0$, then

$$\begin{aligned} f(S') &= \textstyle\sum_{i \in S'} y'_i + \max_{X \subseteq \cup_{i \in N \cup \{0\}} A_i} h(y', S', X) \\ &= \textstyle\sum_{i \in S'} y'_i + h(y', S', X') > \sum_{i \in S} y'_i. \quad \square \end{aligned}$$

5.5.5 Waste Disposal Game

Consider a set $N = \{1, \ldots, n\}$ of firms (players), each of which is involved in production on the shore of a common lake. Each firm i generates w_i units of waste for disposal. Each firm i can determine an amount x_i of its waste to treat (either by hauling this amount of waste away from the lake or by cleaning it satisfactorily and then dumping it into the lake) so that these x_i units of waste do not pollute the lake, and the remaining $w_i - x_i$ units of waste from firm i are dumped untreated into the lake and are a source of pollution in the lake. The waste treatment level x_i of each firm i must be contained in $L_i = [0, w_i]$. There is a cost $b_i(x_i)$ to firm i for treating x_i units of its waste, where $b_i(0) = 0$. There is a cost $c_i(z)$ to each firm i to purify the water that it uses from the lake for its production processes if there are z total units of untreated waste dumped into the lake by the various firms. Each firm i receives a net return r_i from its operations, not including the cost of treating its waste and the cost of purifying water that it uses from the lake. Any coalition that forms optimizes the amounts of waste treated for all members of the coalition and assumes that no waste is treated by any firm not in the coalition. The present model is a generalization of a model of Shapley and Shubik [1969], which further assumes that each waste treatment cost function $b_i(x_i)$ and each water purification cost function $c_i(z)$ is linear in its argument and that certain inequalities involving the slopes of these cost functions hold. Shapley and Shubik [1969] note that their model is a convex game. Let $x = (x_1, \ldots, x_n)$ be the vector composed of the amounts of waste treated by each of the firms in N, let $L = \times_{i \in N} L_i$ be the set of feasible values for x given a coalition consisting of all firms N, and let $X(S) = \{x : x \in L, x_i = 0 \text{ for each } i \in N \setminus S\}$ be the set of feasible values for x given a coalition consisting of any subset S of the firms N. For x in L and any subset S of N, let

$$g(x, S) = \textstyle\sum_{i \in S} r_i - \sum_{i \in N} b_i(x_i) + \sum_{i \in N \setminus S} b_i(0) - \sum_{i \in S} c_i\big(\sum_{i \in N} w_i - \sum_{i \in N} x_i\big).$$

Assume that $g(x, S)$ is bounded above on $L \times \mathcal{P}(N)$. For any subset S of N, the characteristic function is

$$\begin{aligned} f(S) &= \sup\big\{\textstyle\sum_{i \in S} r_i - \sum_{i \in S} b_i(x_i) - \sum_{i \in S} c_i\big(\sum_{i \in N} w_i - \sum_{i \in S} x_i\big) : \\ &\qquad x_i \in L_i \text{ for each } i \in S\big\} \\ &= \sup\{\textstyle\sum_{i \in S} r_i - \sum_{i \in N} b_i(x_i) + \sum_{i \in N \setminus S} b_i(0) \\ &\qquad - \textstyle\sum_{i \in S} c_i(\sum_{i \in N} w_i - \sum_{i \in N} x_i) : \\ &\qquad x \in L, x_i = 0 \text{ for each } i \in N \setminus S\} \\ &= \sup_{x \in X(S)} g(x, S). \end{aligned}$$

Then (N, L, g) is an activity optimization game. Suppose also that each $b_i(x_i)$ and each $c_i(z)$ is lower semicontinuous, so the activity optimization game is playable.

Suppose that, in addition, $c_i(z)$ is increasing and concave in z for each firm i, so this is an activity optimization game with complementarity by part (a) of

Lemma 2.6.2, Theorem 2.6.4, Theorem 2.6.2, and part (b) of Lemma 2.6.1. By Theorem 5.4.1, this is a convex game. By Theorem 5.4.2, optimal waste treatment levels of all firms in any coalition S are an increasing function of S.

Now suppose that each return r_i, each waste treatment cost function $b_i(x_i)$, and each water purification cost function $c_i(z)$ depends on a parameter t contained in a partially ordered set T, and denote this dependence by r_i^t, $b_i^t(x_i)$, and $c_i^t(z)$. Assume that r_i^t is increasing in t for each i, $b_i^t(x_i)$ has decreasing differences in (t, x_i) for each i, $b_i^t(0)$ is decreasing in t for each i, $c_i^t(z)$ is decreasing in t for all i and z, and $c_i^t(z)$ has increasing differences in (t, z) for each i, so t is a complementary parameter. Theorem 5.4.3 implies that optimal waste treatment levels for each member of a coalition are an increasing function of the parameter. Corollary 5.4.1 (together with part (a) of Theorem 5.2.4) implies that the extreme points of the core and the Shapley value are increasing functions of the parameter. Assume also that T is a lattice, r_i^t is supermodular in t for each i, $b_i^t(x_i)$ is submodular in t for all i and x_i, $b_i^t(0)$ does not depend on t for each i, and $c_i^t(z)$ is submodular in t for all i and z, so t is a supermodular parameter. By Theorem 5.4.5, the characteristic function for each coalition is a supermodular function of the parameter.

5.6 Games with Complementarities That Are Not Convex Games

An important feature of activity optimization games with complementarity developed in Section 5.4 is that they offer a rich and natural model for economic interaction and decision making by players, where a condition involving complementarity causes the cooperative game to be a convex game and thus have the properties discussed in Section 5.2. In contrast, Subsection 5.6.1 and Subsection 5.6.2 present other cooperative game models that incorporate an optimal decision problem for each coalition and have a complementarity property, but do not satisfy the formal conditions of an activity optimization game with complementarity. Examples show that these cooperative games need not be a convex game and the core may be empty. Subsection 5.6.1 considers a cooperative game model for the design of a network to accommodate multicommodity shipments by multiple users. Subsection 5.6.2 considers a version of a max-min cooperative game model of Von Neumann and Morgenstern [1944].

5.6.1 Network Design Game

The cooperative game model of this subsection involves the design of a network to accommodate multicommodity flows for multiple cooperating users.

Each coalition optimizes the design of a network under conditions implying a kind of increasing returns to scale. Yet Example 5.6.1 shows that this cooperative game need not be a convex game and the core may be empty. An earlier version of this example is given in Topkis [1980], and Sharkey [1982c] presents an example derived from it.

There is a set $N = \{1, \dots, n\}$ of players. The players use a network consisting of a finite set of nodes and a set E of directed edges, with each such edge being between some ordered pair of the nodes. Each player i is interested in shipping $h_i > 0$ units of a distinct commodity i between a specified ordered pair of nodes, with this ordered pair of nodes depending on i, using edges from E. Carrying out this shipment results in a return r_i for player i.

Initially, each edge of E has 0 capacity for accommodating shipments of the players' commodities. There is an investment cost $c_j(x)$ for installing x units of capacity on edge j of E. For each edge j, the capacity cost function $c_j(x)$ is concave and increasing in x on $[0, \infty)$ and $c_j(0) = 0$.

Any subset S of the players may form a coalition. That coalition could construct capacities on the various edges of E to create a capacitated network in which it is possible to simultaneously satisfy the requirements of each player in S. Of all such feasible networks, coalition S chooses one of minimum cost.

The network design problem for each coalition S is a multicommodity network flow problem in an uncapacitated network with concave and increasing costs on each edge. Therefore there exists an optimal solution that is an extreme point flow, so for each player i in S all h_i units of that player's shipment travel on the same path.

Let P_i be the collection of all paths that can be used to ship player i's commodity between the ordered pair of nodes for player i. Assume that each P_i is nonempty. For each edge j in E, let Q_j be the collection of all paths that are composed of edges from E and that include edge j.

The problem for coalition S is to pick a path p_i in P_i for each player i in the coalition S to minimize $\sum_{j \in E} c_j(\sum_{\{i: i \in S, p_i \in Q_j\}} h_i)$. For each subset S of the players, let

$$C(S) = \min_{\{p_i : p_i \in P_i \text{ for each } i \in S\}} \sum_{j \in E} c_j(\sum_{\{i: i \in S, p_i \in Q_j\}} h_i)$$

so the characteristic function is

$$f(S) = \sum_{i \in S} r_i - C(S).$$

The model exhibits a kind of increasing returns to scale because the concavity of the capacity cost on each edge implies a decreasing marginal cost of capacity. One might conjecture that this property would carry over to the characteristic function so the marginal value of a player to a coalition would be an increasing function of the set of players initially in that coalition and so the cooperative game would be a convex game. Example 5.6.1 shows that this conjecture need not be true. Indeed, the core of this cooperative game may be empty.

Example 5.6.1. Consider a network with nodes $1, \dots, 7$ and the set of edges E consisting of directed edges joining each of the following pairs of nodes in each direction: (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1), (2, 7), (4, 7), (6, 7). There is a set $N = \{1, 2, 3\}$ of players and the node pairs corresponding to the shipments of each of the 3 players are (1, 7), (3, 7), and (5, 7) with $h_i = 1$ for each player i. For some real α with $0 < \alpha < 1$, $c_j(x) = x^\alpha$ for each edge j in E.

For this example, one can determine the optimal multicommodity network flows by inspection. In particular,

$$C(\emptyset) = 0,$$

$$C(\{1\}) = C(\{2\}) = C(\{3\}) = 2,$$

$$C(\{1,2\}) = C(\{1,3\}) = C(\{2,3\}) = 2 + 2^\alpha,$$

and

$$C(\{1,2,3\}) = 4 + 2^\alpha.$$

Thus

$$\begin{aligned} &f(\{1,2,3\}) + f(\{1\}) - f(\{1,2\}) - f(\{1,3\}) \\ &\quad = C(\{1,2\}) + C(\{1,3\}) - C(\{1,2,3\}) - C(\{1\}) \\ &\quad = 2^\alpha - 2 \\ &\quad < 0. \end{aligned}$$

Hence $f(S)$ is not supermodular, and the cooperative game is not a convex game.

Suppose that there exists a vector y in the core. Then

$$y_1 + y_2 \geq f(\{1,2\}) = r_1 + r_2 - C(\{1,2\}) = r_1 + r_2 - 2 - 2^\alpha,$$

$$y_1 + y_3 \geq f(\{1,3\}) = r_1 + r_3 - C(\{1,3\}) = r_1 + r_3 - 2 - 2^\alpha,$$

and

$$y_2 + y_3 \geq f(\{2,3\}) = r_2 + r_3 - C(\{2,3\}) = r_2 + r_3 - 2 - 2^\alpha.$$

Adding the above three inequalities,

$$2(y_1 + y_2 + y_3) \geq 2(r_1 + r_2 + r_3) - 3(2 + 2^\alpha)$$

so

$$y_1 + y_2 + y_3 \geq r_1 + r_2 + r_3 - (3/2)(2 + 2^\alpha).$$

But this contradicts

$$\begin{aligned} y_1 + y_2 + y_3 &= f(\{1, 2, 3\}) \\ &= r_1 + r_2 + r_3 - C(\{1, 2, 3\}) \\ &= r_1 + r_2 + r_3 - 4 - 2^\alpha \\ &< r_1 + r_2 + r_3 - (3/2)(2 + 2^\alpha), \end{aligned}$$

so the core is empty.

One extension of the network design game described above Example 5.6.1 would permit a restriction of the path p_i for each player i to an arbitrary specified subset P'_i of P_i. Then the characteristic function would be

$$f'(S) = \sum_{i \in S} r_i - C'(S)$$

where

$$C'(S) = \min_{\{p_i : p_i \in P'_i \text{ for each } i \in S\}} \sum_{j \in E} c_j(\sum_{\{i: i \in S, p_i \in Q_j\}} h_i).$$

In the special case where each P'_i consists of a single path p'_i,

$$f(S) = \sum_{i \in S} r_i - \sum_{j \in E} c_j(\sum_{\{i: i \in S, p'_i \in Q_j\}} h_i)$$

is supermodular as a result of the concavity of each $c_j(x)$, part (a) of Lemma 2.6.2, Theorem 2.6.4, and part (b) of Lemma 2.6.1. Thus the cooperative game is a convex game.

5.6.2 Max-Min Game

Following is a version of a model of Von Neumann and Morgenstern [1944] that determines the characteristic function for each coalition as the value of a certain two-player noncooperative game. There is a set $N = \{1, \ldots, n\}$ of players. Each player i chooses a strategy x_i from a finite set X_i, and the joint strategy is $x = (x_1, \ldots, x_n)$. The net return to player i from a joint strategy x is $g_i(x)$. It is assumed that the noncooperative game is zero-sum, so $\sum_{i \in N} g_i(x) = 0$ for each feasible joint strategy x. The characteristic function $f(S)$ for a coalition S of the players is equal to the value of the following zero-sum two-player noncooperative game. There are two players, denoted I and II, with player I corresponding to the set of players S in the original noncooperative game and player II corresponding to the set of players $N \setminus S$ in the original noncooperative game. The strategies of player I consist of the strategies x_i for each player i in S, and the strategies of player II consist of the strategies x_i for each player i in $N \setminus S$. The strategies of player I and player II thus take the form $\{x_i : i \in S\}$ and $\{x_i : i \in N \setminus S\}$, respectively. Any strategy for player I and any strategy for player II combine into a joint strategy x for the original noncooperative game and result in a return $\sum_{i \in S} g_i(x)$ to player I and a return $\sum_{i \in N \setminus S} g_i(x) = -\sum_{i \in S} g_i(x)$ to player II. The noncooperative game takes the form of a max-min problem where, for each strategy of player I, player II minimizes the return to player I over all strategies of player II, and player I maximizes that quantity over all strategies of player I.

Von Neumann and Morgenstern [1944] show that the characteristic function $f(S)$ is superadditive. However, Example 5.6.2 shows that this model does not appear to generally lead to a supermodular characteristic function even in a case where each player's payoff function is exceptionally well-behaved. Furthermore, the core may be empty.

Example 5.6.2. Let $N = \{1, 2, 3\}$, $X_1 = \{0, 1\}$, $X_2 = X_3 = \{0\}$, $g_1(x) = 0$, $g_2(x) = x_1$, and $g_3(x) = -x_1$. Here, each $g_i(x)$ and each $-g_i(x)$ is concave, convex, supermodular, and submodular on R^3. But $f(\{1\}) = 0$, $f(\{1, 2\}) = 1$, $f(\{1, 3\}) = 0$, and $f(\{1, 2, 3\}) = 0$, so $f(\{1, 2, 3\}) + f(\{1\}) - f(\{1, 2\}) - f(\{1, 3\}) = -1 < 0$. Hence $f(S)$ is not supermodular and the cooperative game is not a convex game.

Suppose that there exists a vector y in the core. Then

$$y_1 + y_2 \geq f(\{1, 2\}) = 1,$$

$$y_1 + y_3 \geq f(\{1, 3\}) = 0,$$

and

$$y_2 + y_3 \geq f(\{2, 3\}) = 0.$$

Adding the above three inequalities, $2(y_1 + y_2 + y_3) \geq 1$. But this contradicts $y_1 + y_2 + y_3 = f(\{1, 2, 3\}) = 0$. Thus, the core is empty.

The characteristic function in Example 5.6.2 would be the same if $X_1 = X_2 = X_3 = [0, 1]$, so it is not crucial to the example that the set of strategies be finite or nonconvex. Furthermore, this example would be unaffected if the strategies of the players were permitted to be mixed strategies rather than pure strategies.

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